SECTION 8

LIGHT-TIME SOLUTION

Contents

Figure

8.1 INTRODUCTION

The first step in obtaining the computed value of an observed quantity is to obtain the light-time solution for that observable. This section describes the spacecraft light-time solution used to obtain the computed values of all spacecraft observables and the quasar light-time solution used to obtain the computed values of narrowband and wideband quasar interferometry observables (described in Section 13). The spacecraft light-time solution can be obtained in the Solar-System barycentric space-time frame of reference or in the local geocentric space-time frame of reference. The Solar-System barycentric frame of reference can be used for a spacecraft located anywhere in the Solar System. The local geocentric frame of reference can be used for a spacecraft that is very near the Earth (*e.g.*, a low Earth orbiter). Note that the Moon is not close enough to the Earth to use this frame of reference, and its motion must be represented in the Solar-System barycentric space-time frame of reference. The quasar light-time solution is obtained in the Solar-System barycentric space-time frame of reference.

Quantities from each spacecraft light-time solution are used to calculate a precision one-way or round-trip light time between a tracking station on Earth (or an Earth satellite) and the spacecraft. Quantities from each quasar light-time solution are used to calculate a precision delay of the quasar wavefront from its reception at receiver 1 to its reception at receiver 2. Either receiver can be a tracking station on Earth or an Earth satellite. These precision light times are calculated from the formulations given in Section 11. The computed value of each observable is obtained from one, two, or four light-time solutions and the corresponding computed precision light times as described in Section 13.

The spacecraft light-time solution produces position, velocity, and acceleration vectors of the receiver at the reception time t_3, the spacecraft at the reflection time t_2 (for round-trip data) or transmission time t_2 (for one-way data), and the transmitter (for round-trip data) at the transmission time t_1. The receiver or the transmitter can be a tracking station on Earth or an Earth satellite. The spacecraft can be a free spacecraft or a landed spacecraft (resting on any celestial body in the Solar System). In the Solar-System barycentric frame of reference, the position, velocity, and acceleration vectors at t_3, t_2, and t_1 are referred to the Solar-System barycenter. In the local geocentric frame of reference, the position, velocity, and acceleration vectors are referred to the center of mass of the Earth.

The quasar light-time solution produces position, velocity, and acceleration vectors of receiver 1 at the reception time t_1 of the quasar wavefront at receiver 1 and position, velocity, and acceleration vectors of receiver 2 at the reception time t_2 of the quasar wavefront at receiver 2. These vectors are referred to the Solar-System barycenter. Either receiver can be a tracking station on Earth or an Earth satellite.

Section 8.2 gives the equations for the position, velocity, and acceleration vectors of the receiver, spacecraft, and transmitter for a spacecraft light-time solution. It also gives the equations for the position, velocity, and acceleration vectors of the two receivers for a quasar light-time solution.

Section 8.3 describes the spacecraft light-time solution. The light-time equation is derived in Section 8.3.1. The differential corrector, which is used in the iterative solution for the epochs t_2 and t_1, is given in Section 8.3.2. The down-leg predictor, which gives the first estimate of the epoch t_2, is given in Section 8.3.3. The up-leg predictor, which gives the first estimate of the epoch t_1, is given in Section 8.3.4. Quantities that are calculated or interpolated at the penultimate estimate for t_2 or t_1 are mapped to the final value of the epoch using the equations given in Section 8.3.5. The algorithm for the spacecraft light-time solution in the Solar-System barycentric or local geocentric frame of reference is given in Section 8.3.6.

Section 8.4 describes the quasar light-time solution. The quasar light-time equation is derived in Section 8.4.1. The differential corrector which is used in determining the reception time t_2 at receiver 2 is given in Section 8.4.2. The algorithm for the quasar light-time solution in the Solar-System barycentric frame of reference is given in Section 8.4.3.

8.2 POSITION, VELOCITY, AND ACCELERATION VECTORS OF PARTICIPANTS

This section gives the high-level equations for the position, velocity, and acceleration vectors of the participants in the spacecraft and quasar light-time solutions. The vectors for a participant are evaluated at the epoch of participation of the participant. The epochs of participation used in this section are the arguments for computing the position, velocity, and acceleration vectors of the participants and are specified in coordinate time (ET) of the Solar-System barycentric space-time frame of reference or the local geocentric space-time frame of reference.

For a spacecraft light-time solution in the Solar-System barycentric space-time frame of reference, the position, velocity, and acceleration vectors of the receiver at the reception time t_3, the spacecraft at the reflection time or transmission time t_2, and the transmitter at the transmission time t_1, all of which are referred to the Solar-System barycenter C, are given by:

$$\mathbf{r}_3^{\mathrm{C}}(t_3) = \mathbf{r}_3^{\mathrm{E}}(t_3) + \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_3) \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}} \tag{8–1}$$

$$\mathbf{r}_2^{\mathrm{C}}(t_2) = \mathbf{r}_2^{\mathrm{B}}(t_2) + \mathbf{r}_{\mathrm{B}}^{\mathrm{P}}(t_2) + \mathbf{r}_{\mathrm{B,P}}^{\mathrm{C}}(t_2) \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}} \tag{8–2}$$

$$\mathbf{r}_1^{\mathrm{C}}(t_1) = \mathbf{r}_1^{\mathrm{E}}(t_1) + \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_1) \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}} \tag{8–3}$$

In Eq. (8–1), if the receiver (point 3) is a tracking station on Earth, the first term on the right-hand side is the geocentric space-fixed position vector of the tracking station (in the Solar-System barycentric frame of reference) calculated from the formulation of Section 5. If the receiver is an Earth satellite, the first term is the geocentric space-fixed position vector of the satellite interpolated from the satellite ephemeris (the PV file for the satellite generated by program PV). When the ODP is operating in the Solar-System barycentric space-time frame of reference, PV files are generated in that frame of reference. The second term of Eq. (8–1) is the position vector of the Earth relative to the Solar-System barycenter, obtained by interpolating the planetary ephemeris (Section 3).

In Eq. (8–2), the spacecraft (point 2) can be a free spacecraft or a landed spacecraft. If the spacecraft is landed, point B is the center of mass of the body that the landed spacecraft is resting upon. If the spacecraft is free, point B is the center of integration for the spacecraft ephemeris (PV file). The position vector of a free spacecraft relative to the center of integration B is obtained by interpolating the spacecraft ephemeris (Section 4). If the spacecraft is landed, the space-fixed position vector of the lander relative to the center of mass of the lander body B is calculated from the formulation of Section 6.

The second term on the right-hand side of Eq. (8–2) is non-zero only if the center of integration B for the ephemeris of a free spacecraft or the body B that a landed spacecraft is resting upon is a satellite or the planet of one of the outer planet systems. For this case, the position, velocity, and acceleration vectors of the satellite or planet B of an outer planet system relative to the center of mass P of the planetary system are interpolated from the satellite ephemeris for the planetary system.

If the spacecraft is free and the center of integration B is the Sun, Mercury, Venus, Earth, the Moon, or an asteroid or comet, the third term of Eq. (8–2) is the position vector of body B relative to the Solar-System barycenter, obtained by interpolating the planetary ephemeris (and the small-body ephemeris which contains the asteroid or comet). If the center of integration is the center of mass of an outer planet system, or the planet or a satellite of that system, the third term of Eq. (8–2) is the position vector of the center of mass P of the planetary system relative to the Solar-System barycenter, obtained by interpolating the planetary ephemeris.

If the spacecraft is landed and the lander body B is Mercury, Venus, the Moon, or an asteroid or comet, the third term of Eq. (8–2) is the position vector of body B relative to the Solar-System barycenter, obtained by interpolating the planetary ephemeris (and the small-body ephemeris which contains the asteroid or comet). If the landed spacecraft is resting upon the planet or a satellite of an outer planet system, the third term of Eq. (8–2) is the position vector of the center of mass P of the planetary system relative to the Solar-System barycenter, obtained by interpolating the planetary ephemeris.

In Eq. (8–3), if the transmitter is a tracking station on Earth, the first term on the right-hand side is the geocentric space-fixed position vector of the tracking station calculated from the formulation of Section 5. If the transmitter is an Earth satellite, the first term is the geocentric space-fixed position vector of the satellite interpolated from the satellite ephemeris. The second term is the position vector of the Earth relative to the Solar-System barycenter, obtained by interpolating the planetary ephemeris.

For a spacecraft light-time solution in the local geocentric space-time frame of reference, the position, velocity, and acceleration vectors of the participants are referred to the center of mass E of the Earth. Hence, Eqs. (8–1) and (8–3) reduce to their first terms. The geocentric space-fixed position vector of a receiving or transmitting tracking station on Earth is calculated from the formulation of Section 5. The only difference from the calculations in the Solar-System barycentric frame is that the relativistic transformation from the geocentric frame to the Solar-System barycentric frame (see Section 5.4.2) is not performed in the local geocentric frame of reference. If the receiver or transmitter is an Earth satellite, the geocentric position vector of the satellite is interpolated from the satellite ephemeris (the PV file for the satellite generated by program PV). When the ODP is operating in the local geocentric space-time frame of reference, PV files are generated in that frame of reference. In the local geocentric frame of reference, Eq. (8–2) reduces to its first term which is the geocentric position vector of the free spacecraft, obtained by interpolating its geocentric ephemeris (PV file).

For a quasar light-time solution, the position, velocity, and acceleration vectors of receiver 1 at the reception time t_1 of the quasar wavefront at receiver 1 are given by Eq. (8–3). The position, velocity, and acceleration vectors of receiver 2 at the reception time t_2 of the quasar wavefront at receiver 2 are given by Eq. (8–3) with each subscript 1 replaced by a 2.

8.3 SPACECRAFT LIGHT-TIME SOLUTION

8.3.1 LIGHT-TIME EQUATION

The light-time equation in the Solar-System barycentric space-time frame of reference is derived in Subsection 8.3.1.1. This equation is converted to the light-time equation in the local geocentric space-time frame of reference in Subsection 8.3.1.2. Each of these sections give the auxiliary equations, which are used in the light-time solution to evaluate the light-time equation. Additional equations are given for calculating auxiliary quantities (*e.g.*, the range rate) on the up and down legs of the light path.

8.3.1.1 Solar-System Barycentric Space-Time Frame of Reference

The equation for the light path and the corresponding light-time equation can be derived from the approximate expression (2–16) for the interval *ds*. The first-order term in the light-time equation is the straight line path length between two points divided by the speed of light *c*. The next approximation accounts for the reduction in the coordinate velocity of light v_c below c due to the gravitational potential of the celestial bodies of the Solar System. In terms of the rectangular coordinates of the light path and coordinate time t in the Solar-System barycentric space-time frame of reference, the coordinate velocity of light is defined to be:

$$v_c{}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \tag{8–4}$$

In Eq. (2–16), the interval *ds* is zero along the light path, and the coordinate velocity of light v_c is given by:

$$v_c = c\left[1 - \frac{(1+\gamma)U}{c^2}\right] \tag{8–5}$$

where all terms have been retained to order $1/c^2$, and the gravitational potential U is given by Eq. (2–17). The relativistic light-time delay due to each body of the Solar System accounts for the increase in the light time due to the reduction in v_c below c due to the mass of the body. Since U is linear in the contributions due to the Solar-System bodies and because the velocities of these bodies are small relative to c, the relativistic light-time delay due to each Solar-System body is calculated in its own space-time frame of reference from the one-body metric for that body. Simplifying Eq. (2–16) to the case of one celestial body located at the origin of coordinates, deleting the scale factor l which does not affect the motion of light, and changing to spherical coordinates gives the following expression for the one-body metric, which contains terms to order $1/c^2$ in the components of the metric tensor:

$$ds^2 = \left(1 - \frac{2\mu}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2\gamma\mu}{c^2 r}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \tag{8–6}$$

where r is the radial coordinate, θ is the angle from the z axis, and the angle ϕ is measured from the x axis toward the y axis. The quantity μ is the gravitational constant of the celestial body located at the origin of coordinates. Note that if all terms were retained to order $1/c^2$ in Eq. (8–6), the first parentheses in Eq. (8–6) would contain the additional term $+2\beta\mu^2/c^4 r^2$.

Eq. (8–6) will be used to derive the relativistic light-time delay due to the Sun. This relativistic correction to the Newtonian light time accounts for the reduction in the coordinate velocity of light v_c below c and approximately for the bending of the light path. This same term without the bending effect will be used for calculating the relativistic light-time delay for other Solar-System bodies. The Newtonian light time is the straight-line path length between the transmitter and receiver divided by the speed of light c. It is calculated in the Solar-System barycentric space-time frame of reference in the Solar-System barycentric light-time solution.

The equations of motion for light are the equations of a geodesic curve plus the additional condition that the interval ds is zero along the light path. A geodesic curve extremizes the integral of ds between two points:

$$\delta \int ds = 0 \tag{8–7}$$

We can express this integral as:

$$\delta \int \pounds ds = 0 \tag{8–8}$$

where the Lagrangian £ is given by:

$$\pounds = \frac{ds}{ds} = 1 \tag{8–9}$$

The Euler-Lagrange equations which extremize the integral (8–8) are given by:

$$\frac{d}{ds}\left[\frac{\partial \pounds}{\partial\left(\frac{dq}{ds}\right)}\right] - \frac{\partial \pounds}{\partial q} = 0 \tag{8–10}$$

where $q = r,\ \theta,\ \phi,$ or t. From Eqs. (8–6) and (8–9),

$$\pounds^2 = \left(1 - \frac{2\mu}{c^2 r}\right) c^2 \left(\frac{dt}{ds}\right)^2 - \left(1 + \frac{2\gamma\mu}{c^2 r}\right)\left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2\right] \tag{8–11}$$

Evaluating Eq. (8–10) for $q = \theta$ using Eqs. (8–11) and (8–9) gives:

$$r\frac{d^2\theta}{ds^2} + 2\frac{dr}{ds}\frac{d\theta}{ds}\left(1 - \frac{\gamma\mu}{c^2 r}\right) - r\left(\frac{d\phi}{ds}\right)^2 \sin\theta \cos\theta = 0 \tag{8–12}$$

If coordinates are chosen so that a particle moves initially in the plane $\theta = \pi/2$, $d\theta/ds$ will be zero and Eq. (8–12) gives the result that $d^2\theta/ds^2 = 0$. Thus, in the

1-body problem, the motion of particles and light is planar, and the equations may be simplified by setting

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ \frac{d\theta}{ds} &= 0 \end{aligned} \tag{8–13}$$

Since Eq. (8–11) is explicitly independent of t and ϕ, first integrals of Eq. (8–10) for $q = t$ and ϕ are given by $\partial £ / \partial \left(dt/ds \right) = \text{constant}$ and $\partial £ / \partial \left(d\phi/ds \right) = \text{constant}$. Differentiating Eq. (8–11) accordingly and using Eqs. (8–9) and (8–13) gives:

$$\frac{dt}{ds} = \frac{\text{constant}}{1 - \frac{2\mu}{c^2 r}} \tag{8–14}$$

and

$$\frac{d\phi}{ds} = \frac{\text{constant}}{r^2 \left(1 + \frac{2\gamma\mu}{c^2 r} \right)} \tag{8–15}$$

Dividing Eq. (8–14) by Eq. (8–15) and ignoring $1/c^4$ terms gives:

$$\frac{dt}{d\phi} = r^2 \left[1 + \frac{2(1+\gamma)\mu}{c^2 r} \right] \text{constant} \tag{8–16}$$

Setting $ds = 0$ in Eq. (8–6) and substituting Eq. (8–13) gives:

$$\left(1 - \frac{2\mu}{c^2 r} \right) c^2 dt^2 = \left(1 + \frac{2\gamma\mu}{c^2 r} \right) \left(dr^2 + r^2 d\phi^2 \right) \tag{8–17}$$

Substituting dt from Eq. (8–16) into Eq. (8–17), setting $dr/d\phi = 0$ when $r = R$ (the minimum value of r on the light path), and ignoring $1/c^4$ terms gives:

$$d\phi = \pm \frac{\left[R + \frac{(1+\gamma)\mu}{c^2}\right] dr}{r\left[r^2 + \frac{2(1+\gamma)\mu}{c^2} r - \left(R^2 + \frac{2(1+\gamma)\mu}{c^2} R\right)\right]^{\frac{1}{2}}} \tag{8–18}$$

Integrating between limits of (r, ϕ) and (R, 0) and ignoring $1/c^4$ terms gives:

$$\begin{aligned} \phi &= \pm \left\{ \frac{\pi}{2} - \sin^{-1}\left[\frac{R + \frac{(1+\gamma)\mu}{c^2}}{r} - \frac{(1+\gamma)\mu}{c^2 R} \right] \right\} \\ &= \pm \cos^{-1}\left[\frac{R + \frac{(1+\gamma)\mu}{c^2}}{r} - \frac{(1+\gamma)\mu}{c^2 R} \right] \end{aligned} \tag{8–19}$$

where the plus sign applies for increasing r and the minus sign applies for decreasing r. From the first form of Eq. (8–19), when r approaches ∞, the angle ϕ will approach one of the two asymptotic values:

$$\phi = \pm \left[\frac{\pi}{2} + \frac{(1+\gamma)\mu}{c^2 R} \right] \tag{8–20}$$

The angle between the incoming and outgoing asymptotes is thus:

$$\Delta\phi = \frac{2(1+\gamma)\mu}{c^2 R} \tag{8–21}$$

For general relativity, $\gamma = 1$ and the bending of light $\Delta\phi$ has a maximum value of 8.48 μrad (1.75 arc seconds) when R is equal to the radius of the Sun, 696,000 km. Figure 8–1 shows the curved path of a photon passing the Sun S. Light is moving in the positive y direction, and the point of closest approach occurs at $x = R$, $y = 0$. The polar coordinates (r, ϕ) and rectangular coordinates (x, y) of two points on the light path are shown along with the straight line path (of length r_{12}) joining these two points. The y intercept, which is equal to about 1096 astronomical units, was obtained from Eq. (8–19) by setting $\cos\phi$ equal to zero. The x intercept of the asymptotes follows from the y intercept and the angle of the asymptote.

Given Eq. (8–19) for the light path derived from the one-body metric, we will now derive the corresponding light-time equation from the one-body metric. Substituting $d\phi$ from Eq. (8–16) into Eq. (8–17), setting $dr/dt = 0$ when r equals its minimum value R, and ignoring $1/c^4$ terms gives:

$$dt = \pm \frac{r\left[1 + \dfrac{(1+\gamma)\mu}{c^2 r}\right]^2 dr}{c\left\{\left[r + \dfrac{(1+\gamma)\mu}{c^2}\right]^2 - \left[R + \dfrac{(1+\gamma)\mu}{c^2}\right]^2\right\}^{\frac{1}{2}}} \tag{8–22}$$

Making the following change of variable:

$$\rho = r + \frac{(1+\gamma)\mu}{c^2} \tag{8–23}$$

$$\rho_0 = R + \frac{(1+\gamma)\mu}{c^2} \tag{8–24}$$

gives, ignoring $1/c^4$ terms:

$$dt = \pm \frac{\left[\rho + \dfrac{(1+\gamma)\mu}{c^2}\right] d\rho}{c\left(\rho^2 - {\rho_0}^2\right)^{\frac{1}{2}}} \tag{8–25}$$

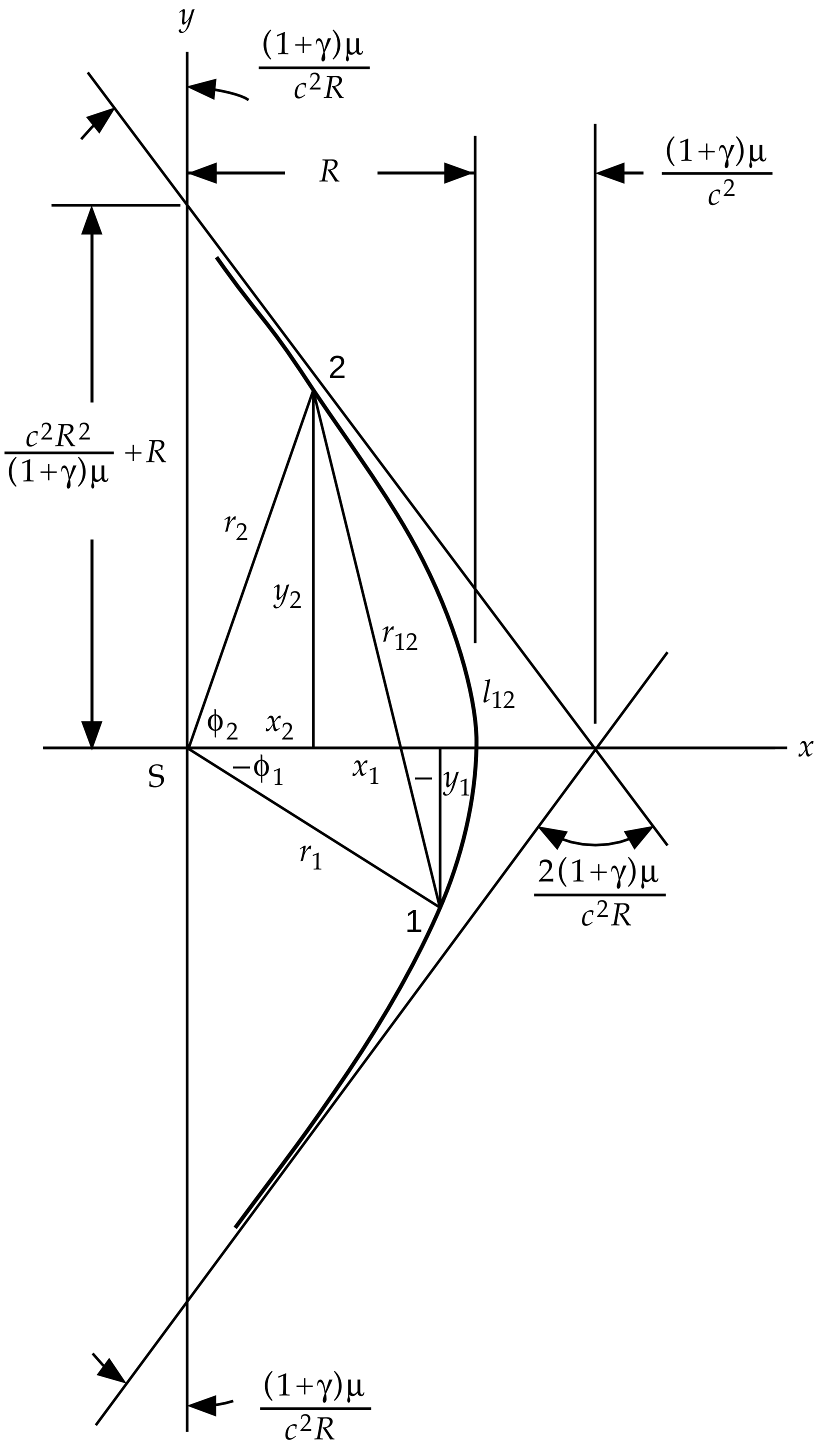

Figure 8–1 Light Path

Expressing the right-hand side as the sum of two terms gives:

$$dt = \pm\frac{1}{c}\frac{\rho\, d\rho}{\left(\rho^2 - {\rho_0}^2\right)^{\frac{1}{2}}} \pm \frac{(1+\gamma)\mu}{c^3}\frac{d\rho}{\left(\rho^2 - {\rho_0}^2\right)^{\frac{1}{2}}} \tag{8–26}$$

Integrating from point 1 (ρ_1, t_1) to point 2 (ρ_2, t_2) gives:

$$\begin{aligned} t_2 - t_1 = &\pm\frac{1}{c}\left[\left({\rho_2}^2 - {\rho_0}^2\right)^{\frac{1}{2}} - \left({\rho_1}^2 - {\rho_0}^2\right)^{\frac{1}{2}}\right] \\ &\pm \frac{(1+\gamma)\mu}{c^3}\ln\left[\frac{\rho_2 + \left({\rho_2}^2 - {\rho_0}^2\right)^{\frac{1}{2}}}{\rho_1 + \left({\rho_1}^2 - {\rho_0}^2\right)^{\frac{1}{2}}}\right] \end{aligned} \tag{8–27}$$

where the plus signs apply when r is strictly increasing from point 1 to point 2, and the minus signs apply when r is strictly decreasing from point 1 to point 2.

At this point, we need a physical interpretation of the quantity $\left(\rho^2 - {\rho_0}^2\right)^{1/2}$. First, let l denote the path length between the points (R, 0) and (r, ϕ):

$$l = \int_{R,0}^{r,\phi}\left(dr^2 + r^2 d\phi^2\right)^{\frac{1}{2}} \tag{8–28}$$

which can be expressed as:

$$l = \int_{R}^{r}\left[1 + r^2\left(\frac{d\phi}{dr}\right)^2\right]^{\frac{1}{2}} dr \tag{8–29}$$

Substituting $d\phi/dr$ from Eq. (8–18), ignoring terms of order $1/c^4$, and substituting Eqs. (8–23) and (8–24) gives:

$$l = \int_{\rho_0}^{\rho} \frac{\rho\, d\rho}{\left(\rho^2 - {\rho_0}^2\right)^{\frac{1}{2}}} \tag{8–30}$$

which is equal to:

$$l = \left(\rho^2 - {\rho_0}^2\right)^{\frac{1}{2}} \tag{8–31}$$

Hence, the quantity $\left(\rho^2 - {\rho_0}^2\right)^{\frac{1}{2}}$ is the path length l between the points $(R, 0)$ and (r, ϕ). We will use the notation:

$$l_2 = \left({\rho_2}^2 - {\rho_0}^2\right)^{\frac{1}{2}} \tag{8–32}$$

and

$$l_1 = \left({\rho_1}^2 - {\rho_0}^2\right)^{\frac{1}{2}} \tag{8–33}$$

We will also denote the path length between any two points 1 and 2 as l_{12}. The next step will be to substitute Eqs. (8–32) and (8–33) into Eq. (8–27) and to transform sums and differences of l_2 and l_1 into the path length l_{12}.

First, we will consider the first term of Eq. (8–27). For r strictly increasing from point 1 to point 2, the first term of Eq. (8–27) is equal to:

$$+\frac{l_2 - l_1}{c} = \frac{l_{12}}{c}$$

For r strictly decreasing from point 1 to point 2, the first term of Eq. (8–27) is equal to:

$$-\frac{l_2 - l_1}{c} = \frac{l_1 - l_2}{c} = \frac{l_{12}}{c}$$

where, in this case, l_1 is greater than l_2. For r decreasing from r_1 to R and then increasing to r_2, the light time from point 1 to point 2 calculated from the first term of Eq. (8–27) is the sum of the first term evaluated on the inbound leg of the light path plus the first term evaluated on the outbound leg of the light path:

$$+\frac{l_2 - 0}{c} - \frac{0 - l_1}{c} = \frac{l_2 + l_1}{c} = \frac{l_{12}}{c}$$

Hence, the first term of Eq. (8–27) can be replaced with the term:

$$\frac{l_{12}}{c} \tag{8–34}$$

where l_{12} is the path length between points 1 and 2.

Now, we will consider the second term of Eq. (8–27). The argument of the natural logarithm can be expressed as:

$$\frac{\rho_2 + \left(\rho_2{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}}{\rho_1 + \left(\rho_1{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}} = \frac{\rho_1 - \left(\rho_1{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}}{\rho_2 - \left(\rho_2{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}} = \frac{\rho_1 + \rho_2 + \left[\left(\rho_2{}^2 - \rho_0{}^2\right)^{\frac{1}{2}} - \left(\rho_1{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}\right]}{\rho_1 + \rho_2 - \left[\left(\rho_2{}^2 - \rho_0{}^2\right)^{\frac{1}{2}} - \left(\rho_1{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}\right]} \tag{8–35}$$

where the second form is obtained from the first by multiplying and dividing by:

$$\left[\rho_1 - \left(\rho_1{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}\right]\left[\rho_2 - \left(\rho_2{}^2 - \rho_0{}^2\right)^{\frac{1}{2}}\right]$$

The third form is obtained from the first two forms by adding the numerators and denominators. For r strictly increasing from point 1 to point 2, the argument of the natural logarithm given by Eq. (8–35) becomes:

$$\frac{\rho_1 + \rho_2 + \left(l_2 - l_1\right)}{\rho_1 + \rho_2 - \left(l_2 - l_1\right)} = \frac{\rho_1 + \rho_2 + l_{12}}{\rho_1 + \rho_2 - l_{12}}$$

and the second term of Eq. (8–27) becomes:

$$+\frac{(1+\gamma)\mu}{c^3}\ln\left[\frac{\rho_1+\rho_2+l_{12}}{\rho_1+\rho_2-l_{12}}\right] \tag{8–36}$$

For r strictly decreasing from point 1 to point 2, the second term of Eq. (8–27) is negative. Changing this sign to positive inverts the argument of the natural logarithm, which becomes:

$$\frac{\rho_1+\rho_2-\left[\left(\rho_2{}^2-\rho_0{}^2\right)^{\frac{1}{2}}-\left(\rho_1{}^2-\rho_0{}^2\right)^{\frac{1}{2}}\right]}{\rho_1+\rho_2+\left[\left(\rho_2{}^2-\rho_0{}^2\right)^{\frac{1}{2}}-\left(\rho_1{}^2-\rho_0{}^2\right)^{\frac{1}{2}}\right]}=\frac{\rho_1+\rho_2-(l_2-l_1)}{\rho_1+\rho_2+(l_2-l_1)}$$

$$=\frac{\rho_1+\rho_2+(l_1-l_2)}{\rho_1+\rho_2-(l_1-l_2)}=\frac{\rho_1+\rho_2+l_{12}}{\rho_1+\rho_2-l_{12}}$$

and the second term of Eq. (8–27) becomes the term (8–36). Note that for this case, l_1 is greater than l_2. For r decreasing from r_1 to R and then increasing to r_2, the light-time correction from point 1 to point 2 calculated from the second term of Eq. (8–27) is the sum of the second term evaluated on the inbound leg of the light path plus the second term evaluated on the outbound leg of the light path. Using the first form of (8–35) for the argument of the natural logarithm, the correction to the light time on the outbound leg is given by:

$$+\frac{(1+\gamma)\mu}{c^3}\left[\ln\left(\rho_2+l_2\right)-\ln\rho_0\right]$$

Using the second form of (8–35) for the argument of the natural logarithm, the correction to the light time on the inbound leg is given by:

$$-\frac{(1+\gamma)\mu}{c^3}\left[\ln\left(\rho_1-l_1\right)-\ln\rho_0\right]$$

The sum of these two terms is:

$$+\frac{(1+\gamma)\mu}{c^3}\ln\left[\frac{\rho_2+l_2}{\rho_1-l_1}\right]$$

Using the same types of procedures used in (8–35), the argument of the natural logarithm can be expressed as:

$$\frac{\rho_2+l_2}{\rho_1-l_1}=\frac{\rho_1+l_1}{\rho_2-l_2}=\frac{\rho_1+\rho_2+(l_1+l_2)}{\rho_1+\rho_2-(l_1+l_2)}=\frac{\rho_1+\rho_2+l_{12}}{\rho_1+\rho_2-l_{12}} \tag{8–37}$$

Hence, for r decreasing from r_1 to R and then increasing to r_2, the effect of the second term of Eq. (8–27) on the light time is given by Eq. (8–36). Since we obtained this same result for r strictly increasing from point 1 to point 2 and also for r strictly decreasing from point 1 to point 2, the second term of Eq. (8–27) can be replaced with the term (8–36).

Replacing the first and second terms of Eq. (8–27) with the terms (8–34) and (8–36) gives the following expression for the one-body light-time equation (where the body is located at the origin of coordinates):

$$t_2-t_1=\frac{l_{12}}{c}+\frac{(1+\gamma)\mu}{c^3}\ln\left[\frac{\rho_1+\rho_2+l_{12}}{\rho_1+\rho_2-l_{12}}\right] \tag{8–38}$$

Figure 8–1 shows the straight-line path (of length r_{12}) between points 1 and 2 and the curved path (of length l_{12}). In order to evaluate Eq. (8–38), we need an approximate expression for $l_{12} - r_{12}$. This expression needs to be reasonably accurate only when the bending of the light path is significant. This only occurs when the transmitter and receiver are on opposite sides of the Sun (the only body for which we consider the bending of the light path). Furthermore, r_1 and r_2 must be large relative to the radius of the Sun, and the minimum radius R (which occurs between r_1 and r_2) must not be large relative to the radius of the Sun. For this geometry, we will assume that light travels along the asymptotes between points 1 and 2. This is a reasonable approximation since the curved light path is much closer to the asymptotes than to the straight-line path connecting points 1 and 2.

In Figure (8–1), let the angle between the straight-line light path (between points 1 and 2) and the inbound asymptote at point 1 (where we assume that the inbound asymptote intersects point 1) be denoted as α_1. Similarly, let the angle between the straight-line path and the outbound asymptote at point 2 (where we assume that the outbound asymptote intersects point 2) be denoted as α_2. The angle between the two asymptotes is $\Delta\phi$ given by Eq. (8–21). Since the sum of the three angles in the triangle formed by the straight-line light path and the two asymptotes is 180 degrees,

$$\alpha_1 + \alpha_2 = \Delta\phi \tag{8–39}$$

For the conditions stated above, the distance D from the straight-line path to the intersection of the asymptotes is given approximately by:

$$y_2\,\alpha_2 = y_1\,\alpha_1 = D \tag{8–40}$$

where we consider y_1 and y_2 to be positive. Solving for α_1 and α_2 gives:

$$\alpha_1 = \Delta\phi\left(\frac{y_2}{y_1 + y_2}\right) \tag{8–41}$$

$$\alpha_2 = \Delta\phi\left(\frac{y_1}{y_1 + y_2}\right) \tag{8–42}$$

and

$$D = \Delta\phi\left(\frac{y_1\,y_2}{y_1 + y_2}\right) \tag{8–43}$$

Given these angles, the approximate expression for $l_{12} - r_{12}$ is given by:

$$
\begin{aligned}
l_{12} - r_{12} &= \frac{y_2}{\cos\alpha_2} + \frac{y_1}{\cos\alpha_1} - y_2 - y_1 \\
&= \frac{y_2}{1 - \frac{\alpha_2{}^2}{2}} + \frac{y_1}{1 - \frac{\alpha_1{}^2}{2}} - y_2 - y_1 \\
&= \frac{y_2\,\alpha_2{}^2}{2} + \frac{y_1\,\alpha_1{}^2}{2}
\end{aligned}
\tag{8–44}
$$

Substituting Eqs. (8–41) and (8–42) into Eq. (8–44) gives:

$$
l_{12} - r_{12} = \frac{(\Delta\phi)^2}{2}\left[\frac{y_1\,y_2}{y_1 + y_2}\right] \tag{8–45}
$$

From Eq. (8–21), this is of order $1/c^4$. If y_1 and y_2 are one astronomical unit, and R is equal to the 696,000 km radius of the Sun, the curved path length l_{12} between points 1 and 2 is 2.7 m longer than the straight-line path length r_{12}. If y_2 approaches infinity, $l_{12} - r_{12}$ approaches 5.4 m. For these same two cases, the values of the distance D between the straight-line path and the intersection of the two asymptotes are 635 km and 1270 km, respectively. From Figure 8–1, the distance between the curved path and the intersection of the asymptotes is about 3 km. Hence, the assumption that the curved path is much closer to the asymptotes than the straight-line path is correct.

Substituting Eq. (8–23) into the second term of Eq. (8–38) gives:

$$
t_2 - t_1 = \frac{l_{12}}{c} + \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_1 + r_2 + l_{12} + \frac{2(1+\gamma)\mu}{c^2}}{r_1 + r_2 - l_{12} + \frac{2(1+\gamma)\mu}{c^2}}\right] \tag{8–46}
$$

The second term of Eq. (8–46) is of order $1/c^3$. The effect of the $1/c^2$ terms in the numerator and denominator of the argument of the natural logarithm is of order $1/c^5$. From Eqs. (8–45) and (8–21), the curved path length l_{12} differs from the straight-line path length r_{12} by terms of order $1/c^4$. In the second term of

Eq. (8–46), this difference would produce terms of order $1/c^7$, which are negligible. Hence, in the second term of Eq. (8–46), we can replace l_{12} with r_{12}:

$$t_2 - t_1 = \frac{l_{12}}{c} + \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_1 + r_2 + r_{12} + \frac{2(1+\gamma)\mu}{c^2}}{r_1 + r_2 - r_{12} + \frac{2(1+\gamma)\mu}{c^2}}\right] \tag{8–47}$$

The first term of Eq. (8–47) is the light time from point 1 to point 2 along the curved path at speed c. The second term is the increase in the light time due to traveling along this path at the coordinate velocity of light (see Eq. 8–5), which is less than c. The effect of the bending of the light path on the second term of Eq. (8–47) is due to the $1/c^2$ terms in the numerator and denominator of the argument of the natural logarithm. However, virtually all of the effect comes from the term in the denominator.

The following derivation will give an approximate expression for the effect of the bending of light on the second term of Eq. (8–47). As stated above, this effect is due to the $1/c^2$ term in the denominator of the argument of the natural logarithm. The expression only needs to be reasonably accurate when the effect of the bending is large. This occurs for the geometry stated after Eq. (8–38). This is the same geometry used to derive Eq. (8–45), which gives the effect of the bending of the light path on the first term of Eq. (8–47). In the second term of Eq. (8–47), the natural logarithm can be expressed as the natural logarithm of the numerator minus the natural logarithm of the denominator. The effect of the latter term on Eq. (8–47) is:

$$-\frac{(1+\gamma)\mu}{c^3} \ln\left(r_1 + r_2 - r_{12} + \frac{2(1+\gamma)\mu}{c^2}\right) \tag{8–48}$$

Differentiating this term gives the effect of the $1/c^2$ term in the argument of the natural logarithm:

$$\frac{-\frac{1}{2c}\left[\frac{2(1+\gamma)\mu}{c^2}\right]^2}{r_1 + r_2 - r_{12}} \tag{8–49}$$

Referring to Figure 8–1, for the conditions stated after Eq. (8–38), the denominator of Eq. (8–49) is given to sufficient accuracy by Eq. (8–44) evaluated with:

$$\alpha_1 = \frac{R}{y_1} \tag{8–50}$$

$$\alpha_2 = \frac{R}{y_2} \tag{8–51}$$

Substituting Eqs. (8–50) and (8–51) into Eq. (8–44) and replacing the denominator of Eq. (8–49) with that result gives:

$$-\frac{1}{c}(\Delta\phi)^2\left[\frac{y_1 y_2}{y_1 + y_2}\right] \tag{8–52}$$

This is the effect of the bending of the light path on the second term of Eq. (8–47). From Eq. (8–45), the effect of the bending of the light path on the first term of Eq. (8–47) is given by:

$$\frac{1}{2c}(\Delta\phi)^2\left[\frac{y_1 y_2}{y_1 + y_2}\right] \tag{8–53}$$

The net of these two effects is one-half of Eq. (8–52). Hence, we can replace l_{12} in the first term of Eq. (8–47) with r_{12} and in the second term of Eq. (8–47) we must change $2(1+\gamma)\mu / c^2$ to $(1+\gamma)\mu / c^2$ in the denominator of the natural logarithm. In order for the modified form of Eq. (8–47) to be consistent with Eq. (8–5) for the coordinate velocity of light, we must make the same change in the numerator of the natural logarithm. The final version of the one-body light-time equation is thus given by:

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_1 + r_2 + r_{12} + \frac{(1+\gamma)\mu}{c^2}}{r_1 + r_2 - r_{12} + \frac{(1+\gamma)\mu}{c^2}}\right] \tag{8–54}$$

Eq. (8–54) was derived from the one-body metric given by Eq. (8–6). Eq. (8–6) was obtained by simplifying Eq. (2–16) to the case of one celestial body located at the origin of coordinates. Eq. (2–16) was obtained from Eq. (2–15) by retaining terms in each component g_{ij} of the metric tensor to order $1/c^2$ only. The neglected $1/c^4$ terms of g_{44} affect the light time by a maximum of about 1 cm/c. The neglected components g_{14}, g_{24}, and g_{34} of the metric tensor produce terms in the coordinate velocity of light (see Eq. 8–5) that are of order $1/c^3$. These neglected terms affect the light time by less than 1 cm/c.

Eq. (8–54) can be used to assemble the final form of the light-time equation used in the light-time solution in the Solar-System barycentric space-time frame of reference. The first term of Eq. (8–54) is evaluated in the Solar-System barycentric frame of reference. It is the time for light to travel from point 1 to point 2 along a straight-line path at the speed of light c. This is the Newtonian light time. The second term of Eq. (8–54) accounts for the reduction in the coordinate velocity of light below c and the bending of the light path. The bending increases the path length but also increases the coordinate velocity of light because the curved light path is further away from the gravitating body than the straight-line path. The net effect of the bending is to decrease the light time by the increase in the path length divided by c. The effects of the bending of the light path are due to the $1/c^2$ terms in the argument of the natural logarithm. The second term of Eq. (8–54) including the bending terms is evaluated for the Sun. This same term without the bending terms is evaluated for every other celestial body of the Solar System (the nine planets and the Moon) that the user "turns on". The final form of the light-time equation in the Solar-System barycentric space-time frame of reference is given by:

$$
\begin{aligned}
t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu_S}{c^3} \ln\left[\frac{r_1^S + r_2^S + r_{12}^S + \frac{(1+\gamma)\mu_S}{c^2}}{r_1^S + r_2^S - r_{12}^S + \frac{(1+\gamma)\mu_S}{c^2}}\right] \\
+ \sum_{B=1}^{10} \frac{(1+\gamma)\mu_B}{c^3} \ln\left[\frac{r_1^B + r_2^B + r_{12}^B}{r_1^B + r_2^B - r_{12}^B}\right]
\end{aligned}
\qquad
\begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix}
\qquad (8\text{–}55)
$$

where μ_S is the gravitational constant of the Sun and μ_B is the gravitational constant of a planet, an outer planet system, or the Moon. In the spacecraft light-time solution, t_1 refers to the transmission time at a tracking station on Earth or at an Earth satellite, and t_2 refers to the reflection time at the spacecraft or, for one-way data, the transmission time at the spacecraft. The reception time at a tracking station on Earth or at an Earth satellite is denoted by t_3. Hence, Eq. (8–55) is the up-leg light-time equation. The corresponding down-leg light-time equation is obtained by replacing 1 with 2 and 2 with 3 as indicated after the equation.

The following equations will be used to evaluate Eq. (8–55) on the up and down legs of the light path in the light-time solution in the Solar-System barycentric space-time frame of reference. Equations are also given for calculating certain auxiliary quantities used at various places in program Regres. The light-time solution in the Solar-System barycentric frame of reference gives the position, velocity, and acceleration vectors referred to the Solar-System barycenter C of the receiver (point 3) at the reception time t_3, the spacecraft (point 2) at the reflection time or transmission time t_2, and the transmitter (point 1) at the transmission time t_1. These vectors, which are calculated from Eqs. (8–1) to (8–3), are denoted as:

$$
\mathbf{r}_3^C(t_3),\ \mathbf{r}_2^C(t_2),\ \mathbf{r}_1^C(t_1) \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}}, \ddot{\mathbf{r}} \qquad (8\text{–}56)
$$

Using these vectors, calculate the following quantities on the up and down legs of the light path:

$$\mathbf{r}_{12} = \mathbf{r}_2^C(t_2) - \mathbf{r}_1^C(t_1) \qquad \begin{matrix}\mathbf{r} \to \dot{\mathbf{r}} \\ 1 \to 2 \\ 2 \to 3\end{matrix} \tag{8–57}$$

$$r_{12} = |\mathbf{r}_{12}| \qquad \begin{matrix}1 \to 2 \\ 2 \to 3\end{matrix} \tag{8–58}$$

where the vertical bars indicate the magnitude of the vector. The range rate on the up and down legs is calculated from:

$$\dot{r}_{12} = \frac{\mathbf{r}_{12}}{r_{12}} \cdot \dot{\mathbf{r}}_{12} \qquad \begin{matrix}1 \to 2 \\ 2 \to 3\end{matrix} \tag{8–59}$$

The following quantities are the negative of the contribution to the range rate on the up and down legs due to the velocity of the transmitter only:

$$\dot{p}_{12} = \frac{\mathbf{r}_{12}}{r_{12}} \cdot \dot{\mathbf{r}}_1^C(t_1) \qquad \begin{matrix}1 \to 2 \\ 2 \to 3\end{matrix} \tag{8–60}$$

Note that r_{12} or r_{23} in the first term of Eq. (8–55) is calculated from Eqs. (8–57) and (8–58). The second term of Eq. (8–55) contains the relativistic delay in the light time due to the Sun S. The third term contains relativistic delays for body B equal to Mercury, Venus, Earth, the Moon, and the planetary systems Mars through Pluto. The delay due to each of these ten bodies can be turned on or off by the user on the GIN file. The following equations can be used to calculate the three variables in the third term of Eq. (8–55) and, when B = the Sun S, the three variables in the second term. For each body B (up to eleven bodies, including the Sun S), the light-time solution interpolates the planetary ephemeris for the position, velocity, and acceleration vectors of body B relative to the Solar-System barycenter C at the epochs of participation t_3, t_2, and t_1:

$$\mathbf{r}_B^C(t_3),\ \mathbf{r}_B^C(t_2),\ \mathbf{r}_B^C(t_1) \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}} \tag{8–61}$$

Calculate the position vector of each participant relative to body B at its epoch of participation:

$$\mathbf{r}_1^B(t_1) = \mathbf{r}_1^C(t_1) - \mathbf{r}_B^C(t_1) \qquad 1 \to 2, 3 \tag{8–62}$$

Using these vectors, calculate the up-leg and down-leg position vector differences relative to body B from:

$$\mathbf{r}_{12}^{\mathrm{B}} = \mathbf{r}_2^{\mathrm{B}}(t_2) - \mathbf{r}_1^{\mathrm{B}}(t_1) \qquad \begin{matrix} 1 \to 2 \\ 2 \to 3 \end{matrix} \tag{8–63}$$

Calculate the magnitudes of the three vectors in Eqs. (8–62):

$$r_1^{\mathrm{B}} = \left| \mathbf{r}_1^{\mathrm{B}}(t_1) \right| \qquad 1 \to 2, 3 \tag{8–64}$$

Calculate the magnitudes of the two vectors in Eqs. (8–63):

$$r_{12}^{\mathrm{B}} = \left| \mathbf{r}_{12}^{\mathrm{B}} \right| \qquad \begin{matrix} 1 \to 2 \\ 2 \to 3 \end{matrix} \tag{8–65}$$

For light passing a celestial body, starting at radius r_1, decreasing to a minimum radius R, and then increasing to radius r_2, the relativistic light-time delay due to the mass of the body (one of the natural logarithm terms of Eq. 8–55) is given approximately by:

$$\frac{(1+\gamma)\mu}{c^3} \ln\left[\left(\frac{2r_1}{R} \right) \left(\frac{2r_2}{R} \right) \right] \tag{8–66}$$

where μ is the gravitational constant of the body. This equation is quite accurate when r_1, r_2 >> R. For light traveling from Jupiter, grazing the surface of the Sun, and arriving at the Earth, r_1 = 5 astronomical units (see Section 4, after Eq. 4–12, for the number of kilometers per astronomical unit), r_2 = 1 astronomical unit, and the radius of the Sun R is 696,000 km. For this case, the relativistic light-time delay due to the mass of the Sun is about 40.6 km/c. For light traveling from Saturn, grazing the surface of Jupiter, and arriving at the Earth, r_1 = r_2 = 5 astronomical units and the radius R of Jupiter is 71,500 km. For this case, the relativistic light-time delay due to the mass of Jupiter is about 56 m/c. For light traveling from Saturn, grazing the surface of the Earth, and then stopping, r_1 = 10 astronomical units and the radius R of the Earth is 6378 km. For this one-way case, the relativistic light-time delay due to the mass of the Earth is calculated from Eq. (8–66) with the factor $2r_2/R$ deleted. The result is a delay of 11.6 cm/c.

In Eq. (8–55), the relativistic light-time delay due to each celestial body of the Solar System is calculated in the space-time frame of reference of that body. The error in the calculated delay due to ignoring the Solar-System barycentric velocity of the gravitating body has an order of magnitude equal to the calculated delay multiplied by the velocity of the body divided by the speed of light c. In the examples given above for the relativistic light-time delays due to the Sun, Jupiter, and the Earth, the errors in the calculated delays due to ignoring the Solar-System barycentric velocities of these bodies are about 2 mm/c, 3 mm/c, and 0.01 mm/c, respectively.

In Eq. (8–55), the relativistic light-time delay due to the Sun accounts for the bending of the light path due to the Sun. However, the relativistic light-time delay due to each other body of the Solar System does not account for the bending of the light path due to that body. The largest error occurs for Jupiter. For a light path starting 5 astronomical units from Jupiter, grazing the surface of Jupiter, and ending 5 astronomical units from Jupiter, the relativistic light-time delay due to the mass of Jupiter is about 56 m/c. The error in this calculation due to ignoring the bending of the light path due to the mass of Jupiter is about 1 mm/c.

Consider a light path between the Earth and a distant spacecraft, which grazes the surfaces of the Sun and Jupiter. The bending of light due to the Sun changes the closest approach radius R at Jupiter and hence the relativistic light-time delay due to Jupiter. Similarly, the bending of light due to Jupiter changes the closest approach radius at the Sun and hence the relativistic light-time delay due to the Sun. Since neither of these effects are included in the light-time equation, the sizes of these effects are errors in Eq. (8–55) for the light time.

First, consider that the transmitter is the Earth, and the light path grazes the surfaces of the Sun and Jupiter on the way to an infinitely distant spacecraft. For this case, r_1 relative to the Sun is 1 astronomical unit, and r_2 relative to the Sun is infinite. The distance D from the straight-line light path to the intersection of the incoming and outgoing asymptotes at the Sun is given by Eq. (8–43), where $\Delta\phi$ is the bending of light due to the Sun, given by Eq. (8–21). For this case, D = 1270 km, and the outgoing asymptote is parallel to the straight-line light path. From Eqs. (8–66) and (8–21), the change in the relativistic light-time delay due to a change ΔR in the closest approach radius R is given by the bending of light $\Delta\phi$ due to the body, calculated from Eq. (8–21), multiplied by $\Delta R/c$. The error in the relativistic light-time delay due to Jupiter due to calculating R from the straight-line light path instead of from the curved path is $\Delta\phi$ for Jupiter (calculated for R = 71500 km) which is 7.887 x 10^{-8} radians multiplied by ΔR = 1270 km/c. The resulting error is 10 cm/c.

Next, consider that the transmitter is a distant spacecraft, and the light path grazes the surfaces of Jupiter and the Sun on the way to the Earth. For this case, r_1 relative to Jupiter is infinite and r_2 relative to Jupiter is 6 astronomical units. The distance D from the straight-line light path to the intersection of the incoming and outgoing asymptotes at Jupiter is given by Eq. (8–43), where $\Delta\phi$ is the bending of light due to Jupiter, given by Eq. (8–21). For this case, D = 71 km, and the outgoing asymptote at Jupiter intersects the Earth. The change in the closest approach radius R at the Sun is 71 km/6 = 11.8 km. The error in the relativistic light-time delay due to the Sun due to calculating R from the straight-line light path instead of from the curved path is $\Delta\phi$ for the Sun (calculated for R = 696000 km), which is 8.486 x 10^{-6} radians multiplied by ΔR = 11.8 km/c. The resulting error is 10 cm/c.

8.3.1.2 Local Geocentric Space-Time Frame of Reference

From Sections 4.5.2 to 4.5.4 and Section 4.4.3, the geometry of space-time near the Earth is described by the one-body point-mass isotropic metric for the Earth in an inertial coordinate system that is rotating due to geodesic precession and the Lense-Thirring precession. The rotation rate of the geocentric inertial coordinate system is about 3 x 10^{-15} rad/s due to geodesic precession and about 2 x 10^{-14} rad/s near the Earth due to the Lense-Thirring precession.

The light-time solution in the local geocentric space-time frame of reference is obtained in a non-inertial frame of reference, which is non-rotating relative to the Solar-System barycentric space-time frame of reference. When formulating the equations of motion in the non-inertial geocentric frame of reference, it must be considered to be rotating with angular velocity – Ω (where

Ω is the rotation rate of the inertial frame of reference). The down-leg light path in the local geocentric frame of reference begins with the correct position of the GPS satellite at the transmission time t_2 and ends with the correct position of the TOPEX satellite or a GPS receiving station on Earth at the reception time t_3. In the non-rotating and non-inertial geocentric frame of reference, the Coriolis and centrifugal accelerations produce a slight curvature of the light path. However, in the local geocentric frame of reference, the light-time solution uses a straight-line light path. Neglect of the curvature of this path produces a negligible error in the light time.

The one-body point-mass metric for the Earth is given by Eq. (4–60). Converting from rectangular to spherical coordinates and retaining terms to order $1/c^2$ in the components of the metric tensor gives Eq. (8–6), which was used to derive the one-body light-time equation, given by Eq. (8–54). In the local geocentric space-time frame of reference, the curvature of the light path due to the mass of the Earth can be ignored and the down-leg light-time equation is given by:

$$t_3 - t_2 = \frac{r_{23}^{\mathrm{E}}}{c} + \frac{(1+\gamma)\mu_{\mathrm{E}}}{c^3} \ln\left[\frac{r_2^{\mathrm{E}} + r_3^{\mathrm{E}} + r_{23}^{\mathrm{E}}}{r_2^{\mathrm{E}} + r_3^{\mathrm{E}} - r_{23}^{\mathrm{E}}}\right] \qquad \begin{matrix} 3 \to 2 \\ 2 \to 1 \end{matrix} \qquad (8\text{–}67)$$

The light-time solution in the local geocentric space-time frame of reference is currently a down-leg light-time solution only, which is all that is required for processing GPS/TOPEX data. If an up leg is ever added to the light-time solution in the geocentric frame of reference, the up-leg light-time equation is obtained from Eq. (8–67) by replacing 3 with 2 and 2 with 1.

The variables in Eq. (8–67) and in the corresponding up-leg light-time equation and certain auxiliary quantities can be calculated from Eqs. (8–56) to Eq. (8–60) and Eq. (8–64). In these equations, the superscripts C and B refer to the Earth E. A round-trip light-time solution in the local geocentric space-time frame of reference would produce the vectors given by Eq. (8–56), except that C refers to the Earth E. These vectors are obtained from Eqs. (8–1) to (8–3) as described in the penultimate paragraph of Section 8.2. The variables calculated from

Eqs. (8–57) to (8–60) have a superscript E in the local geocentric frame of reference.

For a signal transmitted from a GPS satellite to the TOPEX satellite or a GPS receiving station on Earth, the second term of Eq. (8–67) is less than 2 cm divided by the speed of light *c*. Because this term is so small, the gravitational constant of the Earth used in computing it can be the value in the barycentric frame obtained from the planetary ephemeris, or the value in the local geocentric frame of reference calculated from the barycentric value using Eq. (4–25).

8.3.2 LINEAR DIFFERENTIAL CORRECTOR FOR TRANSMISSION TIME ON A LEG OF THE LIGHT PATH

In a spacecraft light-time solution, the reception time at a tracking station on Earth or at an Earth satellite is denoted as t_3. The down-leg light-time solution obtains the transmission time t_2 at the spacecraft (free or landed) by an iterative procedure. Given the converged value of t_2, the up-leg light-time solution obtains the transmission time t_1 at a tracking station on Earth or at an Earth satellite by an iterative procedure.

Let t_j and t_i denote the reception and transmission times for a leg of the light path. For the down leg of the light path, j is 3 and i is 2. For the up leg, j is 2 and i is 1. This section develops a linear differential corrector formula for determining the transmission time t_i. For each estimate of the transmission time t_i, the differential corrector produces a linear differential correction Δt_i to t_i.

In terms of j and i, the light-time equation (8–55) in the barycentric frame and the light-time equation (8–67) in the local geocentric frame can be expressed as:

$$t_j - t_i = \frac{r_{ij}}{c} + RLT_{ij} \qquad (8\text{–}68)$$

where RLT_{ij} is the relativistic light-time delay on the ij leg. In the barycentric frame, it is the sum of the natural logarithm terms on the right-hand side of Eq. (8–55). In the local geocentric frame, it is the natural logarithm term on the right-hand side of Eq. (8–67). In the local geocentric frame, r_{ij} is actually r_{ij}^{E}. For a given estimate of the transmission time t_i, let the function f be the corresponding value of the left-hand side of Eq. (8–68) minus the right-hand side of this equation:

$$f = t_j - t_i - \frac{r_{ij}}{c} - RLT_{ij} \tag{8–69}$$

Holding RLT_{ij} fixed, the partial derivative of f with respect to t_i is given by:

$$\frac{\partial f}{\partial t_i} = -1 + \frac{1}{c}\frac{\mathbf{r}_{ij}}{r_{ij}} \cdot \dot{\mathbf{r}}_i^{\mathrm{C}}\left(t_i\right) = -1 + \frac{\dot{p}_{ij}}{c} \tag{8–70}$$

which was obtained by differentiating Eq. (8–69) and Eqs. (8–56) to (8–58) and then substituting Eq. (8–60). These last four equations are used to calculate the variables in Eq. (8–70). In the local geocentric frame, C in these equations and in Eq. (8–70) refers to the Earth E. The solution of Eq. (8–68) for the transmission time t_i is the value of t_i for which the function f is zero. For a given estimate of t_i, and the corresponding values of f and $\partial f/\partial t_i$, the differential correction to t_i which drives f to zero linearly is given by:

$$f + \frac{\partial f}{\partial t_i}\Delta t_i = 0 \tag{8–71}$$

Solving for Δt_i and substituting Eqs. (8–69) and (8–70) gives the desired equation for the linear differential correction Δt_i to t_i:

$$\Delta t_i = \frac{t_j - t_i - \dfrac{r_{ij}}{c} - RLT_{ij}}{1 - \dfrac{\dot{p}_{ij}}{c}} \tag{8–72}$$

All of the variables in Eq. (8–72) are available from the light-time solution.

8.3.3 DOWN-LEG PREDICTOR FOR TRANSMISSION TIME t_2

The down-leg predictor provides the first estimate of the down-leg light time. Subtracting it from the known reception time t_3 at a tracking station on Earth or at an Earth satellite gives the first estimate of the transmission time t_2 at the spacecraft (a free spacecraft or a landed spacecraft).

Let Δt_3 equal the reception time t_3(ET) in coordinate time ET for the current light-time solution minus the value from the last light-time solution computed for the same spacecraft. For deep space tracking, there is only one spacecraft. However, when processing GPS/TOPEX data, there are multiple GPS satellites. Note that the receiving station on Earth or receiving Earth satellite does not have to be the same for the two light-time solutions. Also, let Δt_2 equal the transmission time t_2(ET) in coordinate time ET for the current light-time solution minus the value from the last light-time solution computed for the same spacecraft.

If the current and previous light-time solutions for the same transmitting spacecraft have the same receiver at t_3, the relation between Δt_2 and Δt_3 is given approximately by:

$$\Delta t_2 = \Delta t_3 \left(1 - \frac{\dot{r}_{23}}{c}\right) \tag{8–73}$$

where $\dot{r}_{23}$ is the down-leg range rate given by Eq. (8–59). If the current and previous receivers are different, Δt_2 computed from Eq. (8–73) will be in error by less than 0.03 seconds. For a typical range rate of 30 km/s, the effect of the $\dot{r}_{23}$ term of Eq. (8–73) on Δt_2 is 0.1 s for a data spacing (Δt_3) of 1000 s.

Let $\mathbf{r}_{2_0}$ and $\dot{\mathbf{r}}_{2_0}$ equal the space-fixed position and velocity vectors of the spacecraft at the transmission time t_2 for the last light-time solution for the same spacecraft. These vectors are relative to the Solar-System barycenter C when computing in that frame of reference and are relative to the Earth E in the local geocentric frame of reference. The predicted position vector of the spacecraft at the transmission time t_2 for the current light-time solution is given approximately by:

$$\mathbf{r}_2 = \mathbf{r}_{2_0} + \dot{\mathbf{r}}_{2_0}\Delta t_2 \tag{8–74}$$

Let $\mathbf{r}_3$ equal the space-fixed position vector of the receiver (tracking station on Earth or Earth satellite) at the reception time t_3 for the current light-time solution. It is referred to the Solar-System barycenter in that frame and to the Earth in the local geocentric frame of reference. Then, the predicted down-leg light time is given by:

$$t_3 - t_2 = \frac{\left|\mathbf{r}_3 - \mathbf{r}_2\right|}{c} \tag{8–75}$$

where the bars denote the magnitude of the vector and c is the speed of light.

From Eqs. (8–73) to (8–75) with typical range rates and velocities of 30 km/s, the effect of the $\dot{r}_{23}/c$ term of Eq. (8–73) on the predicted down-leg light time is about 10^{-8} Δt_3. For a very large data spacing Δt_3 of 10^5 seconds (1.16 days), which is extremely unlikely, this effect is 0.001 s which is negligible. Hence, the $\dot{r}_{23}/c$ term of Eq. (8–73) can be discarded, which gives:

$$\Delta t_2 = \Delta t_3 \tag{8–76}$$

From Eqs. (8–74) to (8–76), the predicted down-leg light time can be computed from:

$$t_3 - t_2 = \frac{\left|\mathbf{r}_3 - \mathbf{r}_{2_0} - \dot{\mathbf{r}}_{2_0}\Delta t_3\right|}{c} \tag{8–77}$$

Subtracting the predicted down-leg light time from the reception time t_3(ET) gives the first estimate for the transmission time t_2(ET) at the spacecraft in coordinate time ET.

The error in the predicted light time is less than the magnitude of the first neglected term in the Taylor series for $\mathbf{r}_2$ evaluated somewhere in the interval Δt_2, divided by c:

$$\delta\left(t_3 - t_2\right) < \frac{a\left(\Delta t_3\right)^2}{2c} \tag{8–78}$$

where a is the acceleration of the spacecraft. The maximum acceleration in the Solar System occurs in a region near the Sun. At 3.3 solar radii from the center of the Sun, the acceleration is 25 m/s^2. This acceleration increases to 274 m/s^2 at the surface of the Sun. Except for this region, where it is unlikely that a spacecraft would survive, the maximum acceleration in the rest of the Solar System is 25 m/s^2 which occurs at the surface of Jupiter. With simultaneous tracking data from several tracking stations, Δt_3 can be positive or negative, and its absolute value can vary from zero to the doppler count time. I presume that when a = 25 m/s^2, the count time and data spacing will not exceed 1000 s. Substituting these values into Eq. (8–78) gives a down-leg predictor error of 0.042 s. Furthermore, I presume that far larger count times will be used with much smaller accelerations, but the product $a(\Delta t_3)^2$ will not exceed 25 x 10^6 m. This will allow count times up to 3160 s when a = 2.5 m/s^2, 10,000 s when a = 0.25 m/s^2, and 31,600 s when a = 0.025 m/s^2. In cruise at one astronomical unit from the Sun, the spacecraft acceleration due to the Sun is 5.9 x 10^{-3} m/s^2, and count times as high as 65,000 s can be used. All of these count times are considerably larger than those currently used, especially the larger count times corresponding to the smaller accelerations. Thus, since all of the above count times correspond to a predictor error of 0.042 s, it is safe to say that the predicted down-leg light time will almost always be accurate to better than 0.1 s. Of course, the predicted down-leg light time for the first light-time solution after a large gap in the data may be very inaccurate. The only consequence of this would be a few extra iterations in the down-leg light-time solution for the transmission time t_2.

Eq. (8–77) for the predicted down-leg light time requires a previous light-time solution for the same spacecraft. Hence, for the first light-time solution for each spacecraft, use a predicted down-leg light time of zero. That is, the first estimate of t_2(ET) will be t_3(ET). For GPS/TOPEX data, an estimated down-leg light time of zero is quite accurate since the actual down-leg light time is less than 0.1 s. For a distant spacecraft, the use of an initial light time of zero will simply result in a few extra iterations for determining t_2 for the first light-time solution.

8.3.4 UP-LEG PREDICTOR FOR TRANSMISSION TIME t_1

The up-leg light time differs from the down-leg light time because of the motion of the Earth between the transmission time t_1 at the transmitting station on Earth or at an Earth satellite and the reception time t_3 at the receiving station on Earth or at an Earth satellite and because of the different geocentric positions of the transmitter and receiver at these two times. The up-leg predictor does not account for the geocentric motion of the transmitter between t_1 and t_3 or the different geocentric positions of separate transmitters and receivers. The resulting error in the predicted up-leg light time is up to the Earth's radius divided by the speed of light or 0.021 seconds. Note that the up-leg and down-leg light times are both based upon the position of the spacecraft at the reflection time t_2.

Let $\dot{r}_\mathrm{E}$ denote the contribution to the down-leg range rate due to the velocity of the Earth:

$$\dot{r}_\mathrm{E} = \frac{\mathbf{r}_{23}}{r_{23}} \cdot \dot{\mathbf{r}}_\mathrm{E}^\mathrm{C}\left(t_3\right) \tag{8–79}$$

where the down-leg unit vector is computed from Eqs. (8–56) to (8–58) and the second vector in (8–79) is the velocity vector of the Earth relative to the Solar-System barycenter at the reception time t_3. Note that in the local geocentric space-time frame of reference, this velocity vector is relative to the Earth E and $\dot{r}_\mathrm{E}$ is zero.

Given the converged down-leg light time $t_3 - t_2$ in coordinate time ET and $\dot{r}_E$ calculated from Eq. (8–79), the predicted up-leg light time is calculated from:

$$t_2 - t_1 = (t_3 - t_2)\left(1 - \frac{2\dot{r}_E}{c}\right) \tag{8–80}$$

Subtracting the predicted up-leg light time from t_2(ET) obtained from the down-leg light-time solution gives the first estimate for the transmission time t_1(ET) at the transmitting station on Earth or an Earth satellite.

The up-leg predictor does not account for the acceleration of the Earth acting from t_1 to t_3. The resulting error in the predicted up-leg light time can be calculated from Eq. (8–78) where a refers to the acceleration of the Earth (6×10^{-6} km/s^2) and Δt_3 refers to the round-trip light time. For a spacecraft range of 50 astronomical units, the round-trip light time is 50,000 s, and the error in the predicted up-leg light time is up to 0.025 s. Considering the above-mentioned error of 0.021 s, the total error in the predicted up-leg light time is less than 0.05 seconds. Note that for a spacecraft range of 50 astronomical units, the $\dot{r}_E$ term of Eq. (8–80) contributes about 5 s to the predicted up-leg light time.

If an up leg is ever added to the light-time solution in the local geocentric space-time frame of reference, Eq. (8–80) applies with $\dot{r}_E = 0$.

8.3.5 MAPPING EQUATIONS

The iterative solution for the transmission time t_i for a given leg of the light path continues until the linear differential correction Δt_i to t_i calculated from Eq. (8–72) is less than the value of the input variable LTCRIT. The nominal value for LTCRIT is 0.1 s. Then, position and velocity vectors and related quantities (which are calculated or are interpolated from planetary, small-body, satellite, and spacecraft ephemerides) are mapped from the estimate t_i of the transmission time to the final value $t_i + \Delta t_i$. The mapping equations, which are used at t_2 and at t_1, are given in Subsection 8.3.5.1. The corresponding analysis, which led to the nominal value of 0.1 s for LTCRIT, is given in Subsection 8.3.5.2.

8.3.5.1 Mapping Equations

Space-fixed position and velocity vectors are mapped using quadratic and linear Taylor series:

$$\mathbf{r}(t_i + \Delta t_i) = \mathbf{r}(t_i) + \dot{\mathbf{r}}(t_i)(\Delta t_i) + \tfrac{1}{2}\ddot{\mathbf{r}}(t_i)(\Delta t_i)^2 \tag{8–81}$$

$$\dot{\mathbf{r}}(t_i + \Delta t_i) = \dot{\mathbf{r}}(t_i) + \ddot{\mathbf{r}}(t_i)(\Delta t_i) \tag{8–82}$$

These equations are used to map space-fixed position and velocity vectors interpolated from the planetary ephemeris and small-body ephemeris at t_2 and t_1 (Section 3.1.2.3), a satellite ephemeris at t_2 (Section 3.2.2.2), a spacecraft ephemeris at t_2, calculated body-centered space-fixed position and velocity vectors of a landed spacecraft at t_2 and a tracking station on Earth at t_1, and the ephemeris of an Earth satellite at t_1.

The 3 x 3 body-fixed to space-fixed transformation matrix T_{E} for the Earth at t_1 and the matrix T_{B} for the body B that a landed spacecraft is resting upon at t_2 are mapped using a quadratic Taylor series:

$$T(t_i + \Delta t_i) = T(t_i) + \dot{T}(t_i)(\Delta t_i) + \tfrac{1}{2}\ddot{T}(t_i)(\Delta t_i)^2 \tag{8–83}$$

True sidereal time θ at t_1 is mapped linearly:

$$\theta(t_i + \Delta t_i) = \theta(t_i) + \dot{\theta}(t_i)\,\Delta t_i \tag{8–84}$$

Some mapping is also performed at the reception time t_3 at a tracking station on Earth or at an Earth satellite. In the algorithms for computing ET – TAI at the reception time t_3 at a tracking station on Earth (Section 7.3.1) and at an Earth satellite (Section 7.3.3), position and velocity vectors are mapped from a preliminary estimate of t_3(ET) to the final value of t_3(ET) (which differ by less than 4 x 10^{-5} s) using Eqs. (7–9) and (7–10), which are equivalent to Eqs. (8–81) and (8–82). In the algorithm in Section 7.3.1, the Earth-fixed to space-fixed transformation matrix T_{E} for the Earth and true sidereal time θ are also mapped from the preliminary estimate of t_3(ET) to the final value using Eqs. (8–83) and (8–84).

8.3.5.2 Nominal Value for Variable LTCRIT

The error in the linear differential correction Δt_i calculated from Eq. (8–72) can be up to:

$$\frac{a\left(\Delta t_i\right)^2}{2c} \tag{8–85}$$

where a is the acceleration of the transmitter for the leg of the light path (the spacecraft at t_2 for the down leg, or a tracking station on Earth or Earth satellite at t_1 for the up leg). The maximum acceleration is that of a free spacecraft. From the paragraph containing Eq. (8–78), the highest acceleration that is likely to be encountered is 25 $\mathrm{m/s^2}$. The mapping equations (8–81) to (8–84) use a differential correction Δt_i up to the value of the variable LTCRIT, whose nominal value is 0.1 s. Hence, from (8–85), differential corrections Δt_i up to 0.1 s will be accurate to at least 0.4 ns. Time in the ODP is measured in seconds past J2000. Time up to 30 years from J2000 will be represented to 10^{-8} s on a 17-decimal-digit computer (the ODP is currently programmed on computers that have a word length greater than 16 decimal digits but less than 17 decimal digits). Hence, the error in Δt_i up to 0.1 s calculated from Eq. (8–72) is less than the last bit of time measured in seconds past J2000 on a 17-decimal-digit machine. So, if Δt_i is less than 0.1 s, $t_i + \Delta t_i$ is the final value of t_i.

Of the four mapping equations, the accuracy of Eq. (8–81) for mapped position vectors is the most critical. Computed values of observed quantities are calculated from accurate and precise values of position vectors of the participants. High-accuracy velocity vectors are not required. The error in mapped position vectors calculated from Eq. (8–81) is up to:

$$\tfrac{1}{6} J\left(\Delta t_i\right)^3 \tag{8–86}$$

where J is the magnitude of the jerk vector for participant i. The jerk vector for a free spacecraft can be much higher than the jerk vector for a tracking station or a landed spacecraft. The highest value likely to be encountered anywhere in the Solar System is 5 x 10^{-4} km/s^3. Hence, from (8–86), position vectors mapped through a time interval Δt_i of up to 0.1 s will have a Taylor series truncation error of up to 10^{-4} m. From the preceding paragraph, the time truncation error on a 17-decimal-digit machine is 10^{-8} s. For a typical velocity of 30 km/s, the corresponding error in position is 3 x 10^{-4} m. Hence, mapping quantities through Δt_i up to LTCRIT = 0.1 s is acceptable because the resulting Taylor series truncation error for position vectors is less than the variation in position vectors due to the time truncation error.

From Section 8.3.3, the first estimate for t_2 will almost always be accurate to better than 0.1 s. From Section 8.3.4, the first estimate for t_1 will always be accurate to better than 0.05 s. From Section 8.3.5.2, quantities which are calculated or interpolated at an estimate for t_i (where i = 2 or 1) can be mapped through Δt_i up to LTCRIT = 0.1 s with negligible error. Hence, in almost all circumstances, quantities need to be calculated or interpolated at only one estimate for t_2 and t_1. However, if the user desires to reduce the Taylor series truncation error in Eq. (8–81) by reducing LTCRIT to a smaller value such as 0.01 s, then quantities would have to be calculated or interpolated at two estimates for t_2 and t_1.

It will be seen in Section 13 that computed values of doppler observables are significantly affected by roundoff errors in time and position. One way to eliminate these errors would be to recode programs PV and Regres in quadruple precision (instead of the current double precision). If this is done, the appropriate value for LTCRIT would be 0.4 x 10^{-3} s.

8.3.6 ALGORITHM FOR SPACECRAFT LIGHT-TIME SOLUTION

If the transmitter is a tracking station on Earth or an Earth satellite, the spacecraft light-time solution contains an up leg and a down leg. However, if the spacecraft is the transmitter, the light-time solution contains a down leg only. The spacecraft can be a free spacecraft or a landed spacecraft on any body in the

Solar System. The receiver or transmitter can be a tracking station on Earth or an Earth satellite. The light-time solution can be obtained in the Solar-System barycentric space-time frame of reference for a spacecraft anywhere in the Solar System. If the spacecraft is very near the Earth (such as in a low Earth orbit), the light-time solution can be obtained in the local geocentric space-time frame of reference.

It will be seen in Section 10.2.3.1 and Section 13 that in order to obtain the computed values of spacecraft data types, one, two, or four light-time solutions are required. The starting point for each spacecraft light-time solution is the reception time t_3(ST) in station atomic time ST at a tracking station on Earth or at an Earth satellite. For a DSN tracking station on Earth, the reception time t_3(ST) is at the station location. The antenna correction, which is calculated after the light-time solution from the formulation of Section 10.5, changes the point of reception from the station location (which is on the primary axis of the antenna) to the secondary axis of the antenna. This is the tracking point of the antenna, to which the actual observables are calibrated. For reception at a GPS tracking station on Earth or at an Earth satellite, the reception time t_3(ST) is at the nominal phase center of the receiving antenna. These phase center locations are calculated as described in Sections 7.3.1 and 7.3.3. For each data type, Sections 11.2.1 and 10.2.3.3.1 give the equations for transforming the data time tag and the count time (if any) for the data point to the reception time $t_3(\mathrm{ST})_\mathrm{R}$ at the receiving electronics for each of its light-time solutions. Subtracting the down-leg delay (defined in Section 11.2) as described in Section 11.2.2 gives the reception time t_3(ST) at the station location or nominal phase center.

The spacecraft light-time solution is obtained by performing the following steps:

1. Transform the reception time t_3(ST) to t_3(TAI) in International Atomic Time TAI. Sections 2.5.1, 2.5.2, and 2.5.3 describe these time transformations for a DSN tracking station on Earth, a GPS receiving station on Earth, and a TOPEX satellite, respectively.

2. Transform the reception time t_3(TAI) to t_3(ET) in coordinate time ET. The algorithm that applies for a tracking station on Earth is given in Section 7.3.1. The algorithm in Section 7.3.3 applies for reception at the TOPEX satellite. Steps 1 and 2 produce the reception time t_3 in all of the time scales along the path from t_3(ST) to t_3(ET) and precision values of the time differences of adjacent epochs (see Figures 2–1 and 2–2). Step 2 also produces all of the space-fixed position (P), velocity (V), and acceleration (A) vectors required at t_3. The P, V, and A vectors interpolated from the planetary ephemeris are described in Section 3.1.2.3.1 in the Solar-System barycentric frame and in Section 3.1.2.3.2 in the local geocentric frame of reference. If the receiver is a tracking station on Earth, geocentric space-fixed P, V, and A vectors of the tracking station are calculated from the formulation of Section 5. If the receiver is an Earth satellite, geocentric space-fixed P, V, and A vectors of the satellite are interpolated from the satellite ephemeris. All quantities obtained in Step 2 are in the Solar-System barycentric or local geocentric space-time frame of reference.

3. (Barycentric Frame Only). Add the geocentric space-fixed P, V, and A vectors of the Earth satellite or the tracking station on Earth to the Solar-System barycentric P, V, and A vectors of the Earth to give the Solar-System barycentric P, V, and A vectors of the receiver at the reception time t_3(ET) (see Eq. 8–1).

4. (Barycentric Frame Only). In Eq. (8–55), for each body B and the Sun S for which we calculate a relativistic light-time delay, calculate the vector and scalar distance from the body to the receiver (point 3) at the reception time t_3(ET) from Eqs. (8–62) and (8–64) with each subscript 1 changed to a 3.

5. (Geocentric Frame Only). For use in Eq. (8–67), calculate the magnitude of the geocentric space-fixed position vector of the receiver (point 3) at the reception time t_3(ET).

6. For the first light-time solution for each spacecraft, use zero for the predicted down-leg light time. For all light-time solutions after the first one for each spacecraft, calculate the predicted down-leg light time from Eq. (8–77). Note that the vectors in this equation are geocentric in the local geocentric frame of reference. Subtract the predicted down-leg light time from the reception time t_3(ET) to give the first estimate of the transmission time t_2(ET) at the spacecraft.

7. At the estimate for the transmission time t_2(ET), interpolate the planetary ephemeris and small-body ephemeris for the P, V, and A vectors specified in Section 3.1.2.3.1 in the Solar-System barycentric frame of reference and in Section 3.1.2.3.2 in the local geocentric frame of reference.

8. If the data type is one-way doppler (F_1) or a one-way narrowband (*INS*) or wideband (*IWS*) spacecraft interferometry observable and the spacecraft is within the sphere of influence of one of the outer planet systems, or if the center of integration for the ephemeris of a free spacecraft or the body upon which a landed spacecraft is resting is a satellite or the planet of one of the outer planet systems, interpolate the satellite ephemeris for this planetary system at the estimate for the transmission time t_2(ET) for the P, V, and A vectors specified in Section 3.2.2.2.

9. If the spacecraft is free, interpolate the spacecraft ephemeris for the P, V, and A vectors of the spacecraft relative to its center of integration at the estimate for the transmission time t_2(ET). If the spacecraft is a GPS satellite, interpolate its geocentric ephemeris exactly as specified for the TOPEX satellite in Section 7.3.3. The resulting geocentric position vector of the GPS satellite will be the position vector of its nominal phase center.

10. If the spacecraft is landed, calculate the space-fixed P, V, and A vectors of the landed spacecraft relative to the lander body B at the

estimate for the transmission time t_2(ET) from the formulation of Section 6.

11. (Barycentric Frame Only). Using Eq. (8–2), add the P, V, and A vectors obtained in Steps 7 to 10 to give the Solar-System barycentric P, V, and A vectors of the spacecraft at the estimate for the transmission time t_2(ET).

12. (Barycentric Frame Only). In Eq. (8–55), for each body B and the Sun S for which we calculate a relativistic light-time delay, calculate the vector and scalar distance from the body to the spacecraft (point 2) at the transmission time t_2(ET) from Eqs. (8–62) and (8–64) with each subscript 1 changed to a 2.

13. (Geocentric Frame Only). For use in Eq. (8–67), calculate the magnitide of the geocentric space-fixed position vector of the transmitter (point 2) at the transmission time t_2(ET).

14. Calculate vectors, scalars, and the relativistic light time along the down leg from the spacecraft to the receiver. Calculate $\mathbf{r}_{23}$, $\dot{\mathbf{r}}_{23}$, r_{23} (which is r_{23}^{E} in the local geocentric frame), $\dot{r}_{23}$, and $\dot{p}_{23}$ from Eqs. (8–57) to (8–60). In the Solar-System barycentric frame, these quantities are computed from the Solar-System barycentric vectors given by (8–56). In the local geocentric frame, these quantities are computed from the corresponding geocentric vectors (*i.e.*, replace the superscript C with E in 8–56). In the Solar-System barycentric frame of reference, for each body B and the Sun S for which a relativistic light-time delay is computed in Eq. (8–55), calculate $\mathbf{r}_{23}^{\mathrm{B}}$ and r_{23}^{B} from Eqs. (8–63) and (8–65). Note that the vectors in Eq. (8–63) are calculated in Steps 4 and 12. Given all of these quantities, calculate the down-leg relativistic light-time delay RLT_{23}. In the Solar-System barycentric frame, it is the sum of the natural logarithm terms on the right-hand side of Eq. (8–55). In the local geocentric frame, it is the natural logarithm term on the right-hand side of Eq. (8–67).

15. Given t_3(ET) from Step 2, the current estimate for the transmission time t_2(ET) at the spacecraft, and the quantities computed on the down leg of the light path in Step 14, calculate the linear differential correction Δt_2 to t_2(ET) from Eq. (8–72). Add Δt_2 to t_2(ET) to give the next estimate for the transmission time t_2(ET) at the spacecraft.

16. If the absolute value of Δt_2 is less than the value of the input variable LTCRIT, whose nominal value is 0.1 s, proceed to Step 17. Otherwise, go to Step 7. A second parameter which controls the light-time solution is the input variable NOLT (number of light times), whose nominal value is 4. If convergence (*i.e.*, the absolute value of Δt_2 is less than LTCRIT) is not obtained after NOLT passes through Steps 7 to 15, halt the execution of program Regres.

17. Map everything calculated or interpolated at the last estimate of t_2(ET) in Steps 7 to 10 to the final estimate t_2(ET) + Δt_2 using Eqs. (8–81) to (8–84).

18. Using the mapped quantities from Step 17, repeat Steps 11 to 14.

19. For round-trip light-time solutions, time differences are not computed at the reflection time t_2(ET). However, for one-way doppler, time differences are computed at the transmission time t_2(ET). These calculations are performed after the light-time solution using the formulation given in Section 11.4. For GPS/TOPEX observables, time differences are calculated at the transmission time t_2(ET) at the GPS satellite. Transform the transmission time t_2(ET) at the GPS satellite to the other time scales shown in Fig. 2–2 as described in Section 2.5.5. The algorithm for computing the time difference ET – TAI at the GPS satellite is given in Section 7.3.4.

The remainder of this algorithm for the spacecraft light-time solution applies for the up-leg light-time solution. As currently coded, the up-leg light-time solution applies only in the Solar-System barycentric space-time frame of reference. The light-time solution in the local geocentric frame of reference has a down leg only. Also, the transmitter at the transmission time t_1(ET) must be a tracking station on Earth. The following algorithm applies for the up-leg light-time solution in the Solar-System barycentric frame of reference and also in the local geocentric frame of reference. Also, the transmitter may be an Earth satellite.

20. Calculate the predicted up-leg light time from Eqs. (8–79) and (8–80). In Eq. (8–79), the vectors are available from Steps 2 and 14. In the local geocentric frame of reference, replace $\dot{r}_E$ calculated from Eq. (8–79) with zero. In Eq. (8–80), $\left(t_3 - t_2\right)$ is the converged down-leg light time given by the right-hand side of Eq. (8–55) in the barycentric frame and Eq. (8–67) in the local geocentric frame. It is available from Step 14. Subtract the predicted up-leg light time from the converged estimate of t_2(ET) obtained in Step 15 to give the first estimate of the transmission time t_1(ET) at the transmitter (a tracking station on Earth or an Earth satellite).

21. At the estimate for the transmission time t_1(ET), interpolate the planetary ephemeris for the P, V, and A vectors specified in Section 3.1.2.3.1 in the Solar-System barycentric frame of reference and in Section 3.1.2.3.2 in the local geocentric frame of reference.

22. If the transmitter is an Earth satellite, interpolate the satellite ephemeris for the geocentric space-fixed P, V, and A vectors of the satellite at t_1(ET). This may require calculating the offset from the center of mass of the satellite to the nominal location of its phase center as described in Section 7.3.3.

23. If the transmitter is a tracking station on Earth, calculate its geocentric space-fixed P, V, and A vectors at t_1(ET) from the formulation of Section 5.

24. (Barycentric Frame Only). Using Eq. (8–3), add the P, V, and A vectors obtained in Steps 21 to 23 to give the Solar-System barycentric P, V, and A vectors of the transmitter (a tracking station

on Earth or an Earth satellite) at the estimate for the transmission time t_1(ET).

25. (Barycentric Frame Only). In Eq. (8–55), for each body B and the Sun S for which we calculate a relativistic light-time delay, calculate the vector and scalar distance from the body to the transmitter (point 1) at the transmission time t_1(ET) from Eqs. (8–62) and (8–64).

26. (Geocentric Frame Only). For use in Eq. (8–67), calculate the magnitude of the geocentric space-fixed position vector of the transmitter (point 1) at the transmission time t_1(ET).

27. Calculate vectors, scalars, and the relativistic light time along the up leg from the transmitter to the spacecraft. Calculate $\mathbf{r}_{12}$, $\dot{\mathbf{r}}_{12}$, r_{12} (which is r_{12}^{E} in the local geocentric frame), $\dot{r}_{12}$, and $\dot{p}_{12}$ from Eqs. (8–57) to (8–60). In the Solar-System barycentric frame, these quantities are computed from the Solar-System barycentric vectors given by (8–56). In the local geocentric frame, these quantities are computed from the corresponding geocentric vectors (*i.e.*, replace the superscript C with E in 8–56). In the Solar-System barycentric frame of reference, for each body B and the Sun S for which a relativistic light-time delay is computed in Eq. (8–55), calculate $\mathbf{r}_{12}^{\mathrm{B}}$ and r_{12}^{B} from Eqs. (8–63) and (8–65). Note that the vectors in Eq. (8–63) are calculated in Steps 12 and 25. Given all of these quantities, calculate the up-leg relativistic light-time delay RLT_{12}. In the Solar-System barycentric frame, it is the sum of the natural logarithm terms on the right-hand side of Eq. (8–55). In the local geocentric frame, it is the natural logarithm term on the right-hand side of Eq. (8–67).

28. Given t_2(ET) from Step 15, the current estimate for the transmission time t_1(ET) at the transmitter (a tracking station on Earth or an Earth satellite), and the quantities computed on the up leg of the light path in Step 27, calculate the linear differential correction Δt_1 to t_1(ET)

from Eq. (8–72). Add Δt_1 to t_1(ET) to give the next estimate for the transmission time t_1(ET) at the transmitter.

29. If the absolute value of Δt_1 is less than the value of the input variable LTCRIT, whose nominal value is 0.1 s, proceed to Step 30. Otherwise, go to Step 21. If convergence (*i.e.*, the absolute value of Δt_1 is less than LTCRIT) is not obtained after NOLT passes through Steps 21 to 28, halt the execution of program Regres.

30. Map everything calculated or interpolated at the last estimate of t_1(ET) in Steps 21 to 23 to the final estimate t_1(ET) + Δt_1 using Eqs. (8–81) to (8–84).

31. Using the mapped quantities from Step 30, repeat Steps 24 to 27.

32. If the transmitter is a DSN tracking station on Earth, transform the transmission time t_1(ET) to the other time scales shown in Figure 2–1 as described in Section 2.5.4. The algorithm for computing the time difference ET − TAI at the tracking station on Earth is given in Section 7.3.2. If the transmitter is an Earth satellite, transform t_1(ET) to t_1(ST) as described in Section 2.5.5 with t_2 replaced with t_1 (see Figure 2–2). The algorithm for computing the time difference ET − TAI at the Earth satellite is given in Section 7.3.4 (with t_2 replaced with t_1).

8.4 QUASAR LIGHT-TIME SOLUTION

8.4.1 LIGHT-TIME EQUATION

The spacecraft light-time equation (8–55) in the Solar-System barycentric space-time frame of reference will be modified to apply for light traveling from a distant quasar to a tracking station on Earth or an Earth satellite. Applying this equation to two different receivers (where either receiver can be a tracking station on Earth or an Earth satellite) and then subtracting analytically gives the time for the quasar wavefront to travel from receiver 1 at the reception time t_1(ET) in coordinate time ET to receiver 2 at the reception time t_2(ET).

In the following, consider that the index 1 in Eq. (8–55) is replaced by i, which refers to the quasar, and the index 2 in this equation is replaced by j, which refers to a tracking station on Earth or an Earth satellite. Let r denote the enormous distance from the Solar-System barycenter to the quasar. Then, the distance r_{ij} from the quasar at time t_i to the tracking station on Earth or Earth satellite at time t_j is given by:

$$r_{ij} = r - \mathbf{r}_j^{\mathrm{C}}\left(t_j\right) \cdot \mathbf{L}_{\mathrm{Q}} \tag{8–87}$$

where $\mathbf{r}_j^{\mathrm{C}}\left(t_j\right)$ is the position vector of tracking station or Earth satellite j at the reception time t_j relative to the Solar-System barycenter C and $\mathbf{L}_{\mathrm{Q}}$ is the unit vector from the Solar-System barycenter to the quasar. In Eq. (8–55), consider the relativistic light-time delay due to a specific body B (or the Sun S) and consider the triangle which involves the receiving station on Earth or Earth satellite j , body B, and the distant quasar i. Considering the enormous distance r to the quasar, the numerator of the argument of the natural logarithm in the relativistic light-time delay can be approximated by:

$$r_i^{\mathrm{B}} + r_j^{\mathrm{B}} + r_{ij}^{\mathrm{B}} = 2r \tag{8–88}$$

Considering the above-mentioned subtraction which is to follow, this is an excellent approximation. In the j–B–i triangle, the B–i and j–i sides can be considered to be parallel due to the enormous distance r to the quasar. Then, the denominator of the argument of the natural logarithm in the relativistic light-time delay can be approximated by:

$$r_i^{\mathrm{B}} + r_j^{\mathrm{B}} - r_{ij}^{\mathrm{B}} = r_j^{\mathrm{B}} + \mathbf{r}_j^{\mathrm{B}}\left(t_j\right) \cdot \mathbf{L}_{\mathrm{Q}} \tag{8–89}$$

Substituting Eqs. (8–87) to (8–89) into Eq. (8–55) (with 1 and 2 replaced with i and j) gives the light time from the quasar (point i at time t_i) to a tracking station on Earth or Earth satellite (point j at time t_j):

$$
\begin{aligned}
t_j - t_i = {} & \frac{r}{c} - \frac{1}{c}\mathbf{r}_j^{\mathrm{C}}\left(t_j\right)\cdot\mathbf{L}_{\mathrm{Q}} \\
& + \frac{(1+\gamma)\mu_{\mathrm{S}}}{c^3}\ln\left[\frac{2r}{r_j^{\mathrm{S}} + \mathbf{r}_j^{\mathrm{S}}\left(t_j\right)\cdot\mathbf{L}_{\mathrm{Q}} + \frac{(1+\gamma)\mu_{\mathrm{S}}}{c^2}}\right] \\
& + \sum_{\mathrm{B}=1}^{10}\frac{(1+\gamma)\mu_{\mathrm{B}}}{c^3}\ln\left[\frac{2r}{r_j^{\mathrm{B}} + \mathbf{r}_j^{\mathrm{B}}\left(t_j\right)\cdot\mathbf{L}_{\mathrm{Q}}}\right]
\end{aligned}
\tag{8–90}
$$

Consider that two photons leave the quasar at time t_i. One photon arrives at receiver 1 (a tracking station on Earth or an Earth satellite) at coordinate time t_1(ET); the second photon arrives at receiver 2 (a tracking station on Earth or an Earth satellite) at coordinate time t_2(ET). The travel times $t_2 - t_i$ and $t_1 - t_i$ are given by Eq. (8–90) with $j = 2$ and 1, respectively. Subtracting $t_1 - t_i$ from $t_2 - t_i$ gives the following expression for the time for the quasar wavefront to travel from receiver 1 to receiver 2:

$$
\begin{aligned}
t_2 - t_1 = {} & \frac{1}{c}\left[\mathbf{r}_1^{\mathrm{C}}(t_1) - \mathbf{r}_2^{\mathrm{C}}(t_2)\right]\cdot\mathbf{L}_{\mathrm{Q}} \\
& + \frac{(1+\gamma)\mu_{\mathrm{S}}}{c^3}\ln\left[\frac{r_1^{\mathrm{S}} + \mathbf{r}_1^{\mathrm{S}}(t_1)\cdot\mathbf{L}_{\mathrm{Q}} + \frac{(1+\gamma)\mu_{\mathrm{S}}}{c^2}}{r_2^{\mathrm{S}} + \mathbf{r}_2^{\mathrm{S}}(t_2)\cdot\mathbf{L}_{\mathrm{Q}} + \frac{(1+\gamma)\mu_{\mathrm{S}}}{c^2}}\right] \\
& + \sum_{\mathrm{B}=1}^{10}\frac{(1+\gamma)\mu_{\mathrm{B}}}{c^3}\ln\left[\frac{r_1^{\mathrm{B}} + \mathbf{r}_1^{\mathrm{B}}(t_1)\cdot\mathbf{L}_{\mathrm{Q}}}{r_2^{\mathrm{B}} + \mathbf{r}_2^{\mathrm{B}}(t_2)\cdot\mathbf{L}_{\mathrm{Q}}}\right]
\end{aligned}
\tag{8–91}
$$

In the first term, the Solar-System barycentric position vectors of the two receivers are calculated from Eq. (8–3) as described in the last paragraph of Section 8.2. The position vectors of receiver 1 at the reception time t_1 and receiver 2 at the reception time t_2 relative to each body B and the Sun S are calculated from Eq. (8–62). The magnitudes of these vectors are given by Eq. (8–64).

The unit vector to the quasar, with rectangular components referred to the radio frame (see Section 3.1.1) is given by:

$$\mathbf{L}_{\mathrm{Q_{RF}}} = \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix} \tag{8–92}$$

where α and δ are the right ascension and declination of the quasar in the radio frame, which are obtained from the GIN file. The unit vector to the quasar, with rectangular components referred to the planetary ephemeris frame, is given by:

$$\mathbf{L}_{\mathrm{Q}} = \left(R_{\mathrm{x}} R_{\mathrm{y}} R_{\mathrm{z}}\right)^{\mathrm{T}} \mathbf{L}_{\mathrm{Q_{RF}}} \tag{8–93}$$

where the frame-tie rotation matrices R_{z}, R_{y}, and R_{x} are given by Eqs. (5–117) to (5–119).

Given the Solar-System barycentric P, V, and A vectors of receiver 1 at the reception time t_1 and receiver 2 at the reception time t_2:

$$\mathbf{r}_1^{\mathrm{C}}(t_1),\ \mathbf{r}_2^{\mathrm{C}}(t_2) \qquad\qquad \mathbf{r} \rightarrow \dot{\mathbf{r}},\ \ddot{\mathbf{r}} \tag{8–94}$$

which are obtained as described after Eq. (8–91), calculate $\mathbf{r}_{12}$ and $\dot{\mathbf{r}}_{12}$ from Eq. (8–57). In Eq. (8–91), we want to denote the first term as r_{12}/c. Hence, from Eq. (8–57), r_{12} is given by:

$$r_{12} = -\,\mathbf{r}_{12} \cdot \mathbf{L}_{\mathrm{Q}} \tag{8–95}$$

and its time derivative is given by:

$$\dot{r}_{12} = -\,\dot{\mathbf{r}}_{12} \cdot \mathbf{L}_{\mathrm{Q}} \tag{8–96}$$

Also, calculate the auxiliary quantity:

$$\dot{p}_{12} = \dot{\mathbf{r}}_2^{\mathrm{C}}(t_2) \cdot \mathbf{L}_{\mathrm{Q}} \tag{8–97}$$

The first term of Eq. (8–91) represents the travel time of the quasar wavefront from receiver 1 to receiver 2 at speed c when the quasar wavefront is perpendicular to the unit vector to the quasar. The natural logarithm term due to body B or the Sun S represents the change in this light time due to the bending of the quasar wavefront due to body B or the Sun S. The maximum effect occurs when the quasar wavefront grazes the surface of the body and then intersects the Earth a large distance past the body. For this geometry, it is easy to show that a natural logarithm term in Eq. (8–91) is equal to the total bending of light due to the body calculated from Eq. (8–21) multiplied by the component of the distance between receivers 1 and 2 which is normal to $\mathbf{L}_Q$, divided by c.

The maximum effects of the masses of the Sun, Jupiter, and Saturn on the travel time of the quasar wavefront between the two receivers, calculated from Eq. (8–91), are about 108 m/c, 100 cm/c, and 36 cm/c, respectively, where c is the speed of light. The maximum effect of the mass of the Earth is about 0.6 cm/c. In the argument of the natural logarithm in the Sun term, the μ_S terms in the numerator and denominator represent the effects of the bending of the light path on the arrival times at receivers 1 and 2. The maximum effect of these bending terms is about 20 cm/c. These bending terms are ignored for the other bodies in the Solar System. For Jupiter and Saturn, the resulting errors are a maximum of about 0.10 cm/c and 0.03 cm/c, respectively. Ignoring the indirect effect of the solar bending on the Jupiter and Saturn effects produces maximum errors of 1.8 cm/c and 0.8 cm/c when the raypath grazes the Sun and Jupiter or Saturn. Similarly, ignoring the indirect effect of the Jupiter and Saturn bending on the solar effect produces maximum errors of 0.18 cm/c and 0.07 cm/c for the same geometry.

8.4.2 LINEAR DIFFERENTIAL CORRECTOR FOR RECEPTION TIME AT RECEIVER 2

In a quasar light-time solution, the reception time of the quasar wavefront at receiver 1 is denoted as t_1. The light-time solution obtains the reception time t_2 of the quasar wavefront at receiver 2 by an iterative procedure. This section develops a linear differential corrector formula for determining the reception time t_2. For each estimate of the reception time t_2, the differential corrector produces a linear differential correction Δt_2 to t_2.

Using Eq. (8–95), the quasar light-time equation (8–91) can be expressed as:

$$t_2 - t_1 = \frac{r_{12}}{c} + RLT_{12} \tag{8–98}$$

where RLT_{12} is the relativistic correction to the light time given by the sum of term 2 plus term 3 of Eq. (8–91). For a given estimate of the reception time $t_2 = t_2(\mathrm{ET})$ at receiver 2, let the function f be the corresponding value of the left-hand side of Eq. (8–98) minus the right-hand side of this equation:

$$f = t_2 - t_1 - \frac{r_{12}}{c} - RLT_{12} \tag{8–99}$$

Holding RLT_{12} fixed, the partial derivative of f with respect to t_2 is given by:

$$\frac{\partial f}{\partial t_2} = 1 + \frac{1}{c}\dot{\mathbf{r}}_2^{\mathrm{C}}(t_2) \cdot \mathbf{L}_{\mathrm{Q}} \tag{8–100}$$

Substituting Eq. (8–97) gives:

$$\frac{\partial f}{\partial t_2} = 1 + \frac{\dot{p}_{12}}{c} \tag{8–101}$$

The solution of Eq. (8–98) for the reception time t_2 is the value of t_2 for which the function f is zero. For a given estimate of t_2, and the corresponding values of f and $\partial f / \partial t_2$, the differential correction to t_2 which drives f to zero linearly is given by:

$$f + \frac{\partial f}{\partial t_2}\Delta t_2 = 0 \tag{8–102}$$

Solving for Δt_2 and substituting Eqs. (8–99) and (8–101) gives the desired equation for the linear differential correction Δt_2 to t_2:

$$\Delta t_2 = -\frac{t_2 - t_1 - \dfrac{r_{12}}{c} - RLT_{12}}{1 + \dfrac{\dot{p}_{12}}{c}} \tag{8–103}$$

All of the variables in Eq. (8–103) are available from the quasar light-time solution.

8.4.3 ALGORITHM FOR QUASAR LIGHT-TIME SOLUTION

Given the reception time t_1(ST) of the quasar wavefront in station atomic time ST at receiver 1, the quasar light-time solution gives the reception time t_2(ST) of the quasar wavefront in station time ST at receiver 2. It will be seen in Section 10.2.3.1 and Section 13 that wideband quasar (*IWQ*) data points have one light-time solution and narrowband quasar (*INQ*) data points have two light-time solutions. The starting point for each quasar light-time solution is the reception time t_1(ST) at a DSN tracking station on Earth or at an Earth satellite. For a DSN tracking station on Earth, the reception time t_1(ST) is at the station location. The antenna correction, which is calculated after the light-time solution from the formulation of Section 10.5, changes the point of reception from the station location (which is on the primary axis of the antenna) to the secondary axis of the antenna (the tracking point). For reception at an Earth satellite, the reception time t_1(ST) is at the nominal phase center of the satellite's receiving antenna (Section 7.3.3) or at the satellite's center of mass. For each quasar data type, Sections 11.2.1 and 10.2.3.3.1 give the equations for transforming the data time tag and the count time (if any) for the data point to the reception time $t_1(\mathrm{ST})_\mathrm{R}$ at the receiving electronics for each of its light-time solutions. Subtracting the down-leg delay at receiver 1 (defined in Section 11.2) as described in Section 11.2.2 gives the reception time t_1(ST) at the station location on Earth or at the nominal phase center or center of mass of the Earth satellite. The quasar light-time solution can only be performed in the Solar-System barycentric space-time frame of reference. Quasar data types cannot be processed in the local geocentric space-time frame of reference.

The quasar light-time solution is obtained by performing the following steps:

1. The starting point for the quasar light-time solution is the reception time t_1(ST) at receiver 1. If receiver 1 is a DSN tracking station on Earth, transform t_1(ST) to t_1(TAI) in International Atomic Time using the algorithm given in Section 2.5.1 (with t_3 replaced with t_1). If receiver 1 is an Earth satellite, transform t_1(ST) to t_1(TAI) using the algorithm given in Section 2.5.3 (with t_3 replaced with t_1).

2. Transform the reception time t_1(TAI) to t_1(ET) in coordinate time ET. The algorithm that applies for a tracking station on Earth is given in Section 7.3.1. The algorithm in Section 7.3.3 applies for reception at an Earth satellite. In these algorithms, replace t_3 with t_1. Steps 1 and 2 produce the reception time t_1 in all of the time scales along the path from t_1(ST) to t_1(ET) and precision values of the time differences of adjacent epochs (see Figures 2–1 and 2–2). Step 2 also produces all of the space-fixed position (P), velocity (V), and acceleration (A) vectors required at t_1. The P, V, and A vectors interpolated from the planetary ephemeris are described in Section 3.1.2.3.1. If the receiver is a tracking station on Earth, geocentric space-fixed P, V, and A vectors of the tracking station are calculated from the formulation of Section 5. If the receiver is an Earth satellite, geocentric space-fixed P, V, and A vectors of the satellite are interpolated from the satellite ephemeris. All quantities obtained in Step 2 are in the Solar-System barycentric space-time frame of reference.

3. Add the geocentric space-fixed P, V, and A vectors of the Earth satellite or the tracking station on Earth to the Solar-System barycentric P, V, and A vectors of the Earth to give the Solar-System barycentric P, V, and A vectors of receiver 1 at the reception time t_1(ET) (see Eq. 8–3).

4. In Eq. (8–91), for each body B and the Sun S for which we calculate a relativistic light-time correction, calculate the vector and scalar distance from the body to receiver 1 at the reception time t_1(ET) from Eqs. (8–62) and (8–64).

5. Set the first estimate of the reception time t_2(ET) of the quasar wavefront at receiver 2 equal to t_1(ET).

6. At the current estimate of the reception time t_2(ET) of the quasar wavefront at receiver 2, interpolate the planetary ephemeris for the P, V, and A vectors specified in Section 3.1.2.3.1. Note that for the first estimate of t_2(ET), which is equal to t_1(ET), these quantities are available from Step 2.

7. At the current estimate of t_2(ET), calculate the geocentric space-fixed P, V, and A vectors of receiver 2. If receiver 2 is a tracking station on Earth, use the formulation of Section 5. If receiver 2 is an Earth satellite, obtain these quantities by interpolating the geocentric satellite ephemeris for receiver 2.

8. Using Eq. (8–3) (with each 1 replaced by a 2), add the P, V, and A vectors obtained in Steps 6 and 7 to give the Solar-System barycentric P, V, and A vectors of receiver 2 at the current estimate of t_2(ET).

9. In Eq. (8–91), for each body B and the Sun S for which we calculate a relativistic light-time correction, calculate the vector and scalar distance from the body to receiver 2 at the reception time t_2(ET) from Eqs. (8–62) and (8–64) with each 1 changed to a 2.

10. At the current estimate of t_2(ET), calculate $\mathbf{r}_{12}$ and $\dot{\mathbf{r}}_{12}$ from Eq. (8–57), r_{12} and $\dot{r}_{12}$ from Eqs. (8–95) and (8–96), and $\dot{p}_{12}$ from Eq. (8–97). Calculate the unit vector $\mathbf{L}_Q$ to the quasar from Eqs. (8–92), (8–93), and (5–117) to (5–119). Calculate the relativistic light-time correction RLT_{12}, which is the sum of term 2 and term 3 of Eq. (8–91).

11. Given t_1(ET) from Step 2, the current estimate for the reception time t_2(ET) at receiver 2, and the quantities r_{12}, RLT_{12}, and $\dot{p}_{12}$ calculated in Step 10, calculate the linear differential correction Δt_2 to t_2(ET) from Eq. (8–103). Add Δt_2 to t_2(ET) to give the next estimate for the reception time t_2(ET) at receiver 2.

12. If the absolute value of Δt_2 is less than the value of the input variable LTCRIT, whose nominal value is 0.1 s, proceed to Step 13. Otherwise, go to Step 6. A second parameter which controls the light-time solution is the input variable NOLT (number of light times), whose nominal value is 4. If convergence (*i.e.*, the absolute value of Δt_2 is less than LTCRIT) is not obtained after NOLT passes through Steps 6 to 11, halt the execution of program Regres.

13. Map everything calculated or interpolated at the last estimate of t_2(ET) in Steps 6 and 7 to the final estimate t_2(ET) + Δt_2 using Eqs. (8–81) to (8–84) with i equal to 2.

14. Using the mapped quantities from Step 13, repeat Steps 8 to 10.

15. If receiver 2 is a DSN tracking station on Earth, transform t_2(ET) to t_2(ST) as described in Section 2.5.4, with t_1 replaced with t_2 (see Figure 2–1). The algorithm for computing the time difference ET − TAI at the tracking station on Earth is given in Section 7.3.2 (with t_1 replaced with t_2). If receiver 2 is an Earth satellite, transform t_2(ET) to t_2(ST) as described in Section 2.5.5 (see Figure 2–2). The algorithm for computing the time difference ET − TAI at the Earth satellite is given in Section 7.3.4.